

Spectrahedral cones generated by rank 1 matrices

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Abstract

Let $\mathcal{S}_+^n \subset \mathcal{S}^n$ be the cone of positive semi-definite matrices as a subset of the vector space of real symmetric $n \times n$ matrices. The intersection of \mathcal{S}_+^n with a linear subspace of \mathcal{S}^n is called a spectrahedral cone. We consider spectrahedral cones K such that every element of K can be represented as a sum of rank 1 matrices in K . We shall call such spectrahedral cones rank one generated (ROG). We show that ROG cones which are linearly isomorphic as convex cones are also isomorphic as linear sections of the positive semi-definite matrix cone, which is not the case for general spectrahedral cones. We give many examples of ROG cones and show how to construct new ROG cones from given ones by different procedures. We provide classifications of some subclasses of ROG cones, in particular, we classify all ROG cones for matrix sizes not exceeding 4. Further we prove some results on the structure of ROG cones. We also briefly consider the case of complex or quaternionic matrices. ROG cones are in close relation with the exactness of semi-definite relaxations of quadratically constrained quadratic optimization problems or of relaxations approximating the cone of nonnegative functions in squared functional systems.

Keywords: semi-definite relaxation, exactness, rank 1 extreme ray, quadratically constrained quadratic optimization problem

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1 Introduction

Let \mathcal{S}^n be the real vector space of $n \times n$ real symmetric matrices and $\mathcal{S}_+^n \subset \mathcal{S}^n$ the cone of positive semi-definite matrices. The intersection of the cone \mathcal{S}_+^n with an affine subspace of \mathcal{S}^n is called a *spectrahedron*. Spectrahedra appear as the feasible sets of semi-definite programs and are thus of importance for convex optimization. If the affine subspace happens to be a linear subspace $L \subset \mathcal{S}^n$, then the intersection $K = L \cap \mathcal{S}_+^n$ is a *spectrahedral cone*. The facial structure of spectrahedra and spectrahedral cones has been studied in [18].

The subject of this contribution are spectrahedral cones K satisfying the following property.

Property 1.1. *Every matrix in K can be represented as a sum of rank 1 matrices in K .*

We shall call such spectrahedral cones *rank 1 generated* (ROG). A convex cone in some real vector space will be called a ROG cone if it is linearly isomorphic to a spectrahedral cone possessing Property 1.1. The corresponding linear isomorphism will define a representation of the ROG cone. Clearly the cone \mathcal{S}_+^n itself is ROG.

1.1 Motivation

In this subsection we elaborate on the role ROG cones play in optimization. The main result is that in some commonly arising situations, semi-definite relaxations of non-convex optimization problems are exact if and only if they lead to conic programs over ROG cones.

The condition of being a ROG spectrahedral cone can equivalently be stated in terms of bounded spectrahedra. Namely, the conic hull K of a bounded spectrahedron C not containing the zero matrix is ROG if and only if C is the convex hull of the rank 1 matrices in C . Therefore, if C is a compact section of a ROG spectrahedral cone, then the difficult problem of minimizing a linear function over

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the nonconvex set of rank 1 matrices in C is equivalent to the easy problem of minimizing this linear function over the bounded spectrahedron C .

Property 1.1 is hence in close relation with the exactness of semi-definite relaxations of nonconvex problems in the case when the relaxation is obtained by dropping a rank constraint. Many nonconvex optimization problems which are arising in computational practice fall into this framework, i.e., they can be cast as semi-definite programs with an additional rank constraint. It is this rank constraint which makes the problem nonconvex and difficult to solve. At the same time, dropping the rank constraint provides a convenient way of relaxing the problem to an easily solvable semi-definite program.

A classical example is the MAXCUT problem [7], which can be formulated as the problem of maximizing a linear function over the set of positive semi-definite rank 1 matrices whose diagonal elements all equal 1. By dropping the rank 1 condition, one obtains a semi-definite program which yields an upper bound on the maximum cut. Whether this bound is tight is, however, NP-hard to determine [6].

In this context we shall now consider two applications of ROG spectrahedral cones.

Quadratically constrained quadratic problems. The most general class of problems which can be formulated as semi-definite programs with an additional rank 1 constraint are the quadratically constrained quadratic problems [18],[15]. This class includes also problems with binary decision variables, as the condition $x \in \{a, b\}$ can be cast as the quadratic condition $(x - a)(x - b) = 0$.

A generic quadratically constrained quadratic problem can be written as

$$\min_{x \in \mathbb{R}^n} x^T S x : \quad x^T A_i x = 0, \quad i = 1, \dots, k; \quad x^T B x = 1.$$

Here $A_1, \dots, A_k; B; S$ are real symmetric $n \times n$ matrices defining the homogeneous quadratic constraints, the inhomogeneous quadratic constraint, and the quadratic cost function, respectively. Introducing the matrix variable $X = x x^T \in \mathcal{S}_+^n$, we can write the problem as

$$\min_{X \in K} \langle S, X \rangle : \quad \langle B, X \rangle = 1, \quad \text{rk } X = 1, \quad (1)$$

where $K = L \cap \mathcal{S}_+^n$, and $L \subset \mathcal{S}^n$ is the linear subspace given by $\{X \in \mathcal{S}^n \mid \langle A_i, X \rangle = 0 \quad \forall i = 1, \dots, k\}$. The cone K is hence a linear section of the positive semi-definite matrix cone. Problem (1) can be relaxed to a semi-definite program by dropping the rank constraint,

$$\min_{X \in K} \langle S, X \rangle : \quad \langle B, X \rangle = 1. \quad (2)$$

Naturally, the question arises when the semi-definite relaxation (2) obtained from the nonconvex problem (1) is *exact*, i.e., yields the same optimal value as (1). In general, this question is NP-hard [18]. However, a simple sufficient condition on the spectrahedral cone K is Property 1.1.

Lemma 1.2. *Let the linear subspace $L \subset \mathcal{S}^n$ defined above be such that the cone $K = L \cap \mathcal{S}_+^n$ is rank 1 generated. Then either problems (1),(2) are both infeasible, or problem (2) is unbounded, or problems (1),(2) have the same optimal value.*

Proof. Define the spectrahedron $C = \{X \in K \mid \langle B, X \rangle = 1\}$. Then the feasible set of problem (2) is C , while that of problem (1) is $C_1 = \{X \in C \mid \text{rk } X = 1\}$. If $C = \emptyset$, then both problems are infeasible. Assume that $C \neq \emptyset$. Then $K \neq \{0\}$, and by Property 1.1 every extreme ray of the cone K is generated by a rank 1 matrix. If problem (2) is bounded, then its optimal value is achieved at an extreme point $X \in C$. Since X generates an extreme ray of K , we must have $\text{rk } X = 1$. Thus X is feasible also for problem (1), and the optimal value of (1) is not greater than that of (2). But $C_1 \subset C$, and hence the optimal value of (1) is not smaller than that of (2). Therefore both optimal values must coincide. \square

In particular, if the spectrahedron C is bounded, then problems (1) and (2) are equivalent under the conditions of Lemma 1.2.

Squared functional systems. Another motivation for the study of ROG spectrahedral cones comes from squared functional systems [16]. Let Δ be an arbitrary set and F an n -dimensional real vector space of real-valued functions on Δ . Choose basis functions $u_1, \dots, u_n \in F$. The *squared functional system* generated by these basis functions is the set $\{u_i u_j \mid i, j = 1, \dots, n\}$ of product functions.

This system spans another real vector space V of real-valued functions on Δ . Clearly V does not depend on the choice of the basis functions u_i , since it is also the linear span of the squares f^2 , $f \in F$.

Let us define a linear map $\Lambda : V^* \rightarrow \mathcal{S}^n$ and its adjoint $\Lambda^* : \mathcal{S}^n \rightarrow V$ by $\Lambda^*(A) = \sum_{i,j=1}^n A_{ij} u_i u_j$. Here the space \mathcal{S}^n is identified with its dual by means of the Frobenius scalar product¹. By definition of V the map Λ^* is surjective, and hence the map Λ is injective.

The *sum of squares* (SOS) cone $\Sigma \subset V$, given by the set of all functions of the form $\sum_{k=1}^N f_k^2$ for $f_1, \dots, f_N \in F$, can be represented as the image $\Lambda^*[\mathcal{S}_+^n]$ of the positive semi-definite matrix cone and has nonempty interior. The dual Σ^* of the SOS cone is given by the set of all dual vectors $w \in V^*$ such that $\Lambda(w) \succeq 0$ [16, Theorem 17.1]. By injectivity of Λ it follows that Σ^* is linearly isomorphic to its image $K = \Lambda[\Sigma^*] \subset \mathcal{S}^n$. This image equals the intersection of \mathcal{S}_+^n with the linear subspace $L = \text{Im } \Lambda$. It follows that Σ^* is linearly isomorphic to a spectrahedral cone.

Let $P \subset V$ be the cone of nonnegative functions in V . Since every sum of squares of real numbers is nonnegative, we have the inclusion $\Sigma \subset P$. It is then interesting to know when the cones P and Σ coincide. The following result shows that the cone K being ROG is a necessary condition.

Lemma 1.3. *Assume above notations. If $P = \Sigma$, then the spectrahedral cone $K = L \cap \mathcal{S}_+^n$ is rank 1 generated.*

Proof. For $x \in \Delta$, define the dual vector $w_x \in V^*$ by $\langle w_x, v \rangle = v(x)$ for all $v \in V$. We first show that for all $x \in \Delta$ the matrix $\Lambda(w_x)$ is contained in the set $K_1 = \{X \in K \mid \text{rk } X \leq 1\}$.

Fix $x \in \Delta$ and define the vector $s \in \mathbb{R}^n$ element-wise by $s_i = u_i(x)$, $i = 1, \dots, n$. Then we have for all $A \in \mathcal{S}^n$ that

$$\langle \Lambda(w_x), A \rangle = \langle w_x, \Lambda^*(A) \rangle = \sum_{i,j=1}^n A_{ij} \langle w_x, u_i u_j \rangle = \sum_{i,j=1}^n A_{ij} u_i(x) u_j(x) = \langle ss^T, A \rangle.$$

It follows that $\Lambda(w_x) = ss^T$. Hence the rank of $\Lambda(w_x)$ does not exceed 1. Moreover, we have $\Lambda(w_x) \succeq 0$ and $w_x \in V^*$, and therefore $\Lambda(w_x) \in K$. This proves our claim.

For the sake of contradiction, assume now that $K = \Lambda[\Sigma^*]$ is not ROG. Then there exists a dual vector $y \in \Sigma^*$ such that the matrix $\Lambda(y)$ can be strictly separated from the convex hull of K_1 . In other words, there exists $A \in \mathcal{S}^n$ such that $\langle A, \Lambda(y) \rangle < 0$, but $\langle A, X \rangle \geq 0$ for every $X \in K_1$.

Consider the function $q = \Lambda^*(A) \in V$. For every $x \in \Delta$ we have $q(x) = \langle w_x, \Lambda^*(A) \rangle = \langle \Lambda(w_x), A \rangle \geq 0$, because $\Lambda(w_x) \in K_1$. Hence we have $q \in P$. But $\langle q, y \rangle = \langle \Lambda^*(A), y \rangle = \langle A, \Lambda(y) \rangle < 0$, and therefore $y \notin P^*$.

It follows that $P^* \neq \Sigma^*$ and hence $P \neq \Sigma$. This completes the proof. \square

Thus in every squared functional system where the cone of nonnegative functions coincides with the SOS cone Σ , the dual SOS cone Σ^* is linearly isomorphic to a ROG spectrahedral cone. This allows us to construct ROG cones from such squared functional systems. Let us consider two examples.

- The first example is taken from [16, Section 3.1]. Here $\Delta = \mathbb{R}$, and F is the space of all polynomials $q(t)$ of degree not exceeding $n-1$, equipped with the basis of monomials $1, t, \dots, t^{n-1}$. It is well-known that a univariate polynomial $p(t)$ is nonnegative if and only if it is a sum of squares of polynomials $q(t)$ of lower degree. The corresponding ROG cone K is the cone of all Hankel matrices in \mathcal{S}_+^n and has dimension $2n-1$. We shall denote this cone by Han_+^n . This result can be generalized to the space of polynomials $q(t, x)$ on $\Delta = \mathbb{R} \times \mathbb{R}^m$ which are of degree not exceeding $n-1$ in t and homogeneous of degree 1 in $x = (x_1, \dots, x_m)^T$, equipped with the basis $\{x_1, \dots, x_m, tx_1, \dots, tx_m, \dots, t^{n-1}x_1, \dots, t^{n-1}x_m\}$. The sums of squares representability of the corresponding nonnegative polynomials $p(t, x)$ follows from [21]. The corresponding ROG cone K is the cone of all block-Hankel matrices in \mathcal{S}_+^{nm} with $n \times n$ blocks of size $m \times m$ each, and has dimension $\frac{(2n-1)m(m+1)}{2}$. We shall denote this cone by $\text{Han}_+^{n,m}$. Of course, $\text{Han}_+^n = \text{Han}_+^{n,1}$.

- Let $\Delta = \mathbb{R}^3$ and let F be the 6-dimensional space of homogeneous quadratic polynomials on \mathbb{R}^3 , equipped with the basis $x_1^2, x_2^2, x_3^2, x_2x_3, x_1x_3, x_1x_2$. The space V is then the 15-dimensional

¹The reason for defining the operator Λ by virtue of its adjoint Λ^* is to stay in line with the notations in [16]. This definition explicitly uses a basis of the space F . In a coordinate-free definition, the source space of Λ^* should be the space $\text{Sym}^2(F)$ of contravariant symmetric 2-tensors over F , and the operator Λ^* itself should be defined by linear continuation of the map $f \otimes f \mapsto f^2$, $f \in F$.

space of ternary quartics, and in this space the cone of nonnegative polynomials coincides with the SOS cone [12]. The corresponding ROG cone K is given by all matrices in \mathcal{S}_+^6 of the form

$$A = \begin{pmatrix} a_1 & a_6 & a_5 & a_7 & a_{12} & a_{14} \\ a_6 & a_2 & a_4 & a_{15} & a_8 & a_{10} \\ a_5 & a_4 & a_3 & a_{11} & a_{13} & a_9 \\ a_7 & a_{15} & a_{11} & a_4 & a_9 & a_8 \\ a_{12} & a_8 & a_{13} & a_9 & a_5 & a_7 \\ a_{14} & a_{10} & a_9 & a_8 & a_7 & a_6 \end{pmatrix}, \quad a_1, \dots, a_{15} \in \mathbb{R}.$$

1.2 Outlook

In this subsection we summarize the contents of the paper.

In Section 2 we introduce two notions of isomorphisms, a wider one for general convex cones, and the other for spectrahedral cones.

In Section 3 we study fundamental properties of ROG cones. We establish that the minimal polynomial of a ROG cone, when the latter is viewed as an algebraic interior, is determinantal, and the degree of the cone is given by the maximal rank of the matrices it contains (Subsection 3.1). In Subsection 3.2 we study the facial structure of ROG cones and establish that the rank and the Carathéodory number of its elements coincide. In particular, the rank is an invariant of the elements of a ROG cone under linear isomorphisms. In Subsection 3.3 we prove that the geometry of a ROG cone as a conic convex subset of a real vector space determines its representations as ROG spectrahedral cones uniquely up to isomorphism, which is not true for spectrahedral cones in general.

In Section 4 we describe different methods to construct ROG cones of higher degree from ROG cones of lower degree. The most simple way is taking direct sums, which is considered in Subsection 4.1. This leads to the notion of simple ROG cones, which are defined as those not representable as a non-trivial direct sum. In Subsections 4.2, 4.3 we consider two other ways of constructing ROG cones. The second one can be seen as a generalization of taking direct sums.

In Section 5 we consider some examples of ROG cones. In Subsection 5.1 we investigate ROG cones defined by conditions of the type that a subset of entries in the representing matrices vanishes. This class of ROG cones is linked to chordal graphs and has been studied in [1],[17], see also [14] for a generalization to higher matrix ranks. We show that these cones can be constructed from full matrix cones \mathcal{S}_+^k by the methods presented in Section 4. In Subsection 5.2 we construct an example of a continuous family of mutually non-isomorphic ROG cones.

In Section 6 we consider ROG cones of low codimension (Subsections 6.1, 6.2) and simple ROG cones of low dimension (Subsection 6.3).

In Section 7 we consider the variety of extreme rays of ROG cones. We show that the discrete part of this variety factors out and does not interfere with the part corresponding to the continuous components.

In Section 8 we give a complete classification of ROG cones for degrees $n \leq 4$ up to isomorphism.

Finally, we briefly consider the case of complex and quaternionic Hermitian matrices in Section 9.

We conclude the paper with a summary and an outlook on future work.

2 Preliminaries

In this section we formalize some properties of general spectrahedral cones, in particular we rigorously define the notion of isomorphism. Although this notion is widely used implicitly, to our best knowledge it has not yet been explicitly defined in the literature.

2.1 Notions of isomorphism

When studying a class of mathematical objects, one has to distinguish between intrinsic properties of the object and those induced by the often necessary coordinate representation. The intrinsic properties are those which are preserved by the isomorphisms of the class. In this paper, the objects under consideration are spectrahedral cones. It is hence necessary to define when spectrahedral cones are considered isomorphic. We shall consider two different notions of isomorphism. The weaker notion forgets about the matricial nature of spectrahedral cones and considers them just as subsets of a real vector space.

Definition 2.1. Let $K \subset \mathbb{R}^n$, $K' \subset \mathbb{R}^{n'}$ be convex cones. We say that K and K' are *linearly isomorphic* if there exists a bijective linear map $l : \text{span } K \rightarrow \text{span } K'$ such that $l[K] = K'$.

Here the dimensions n, n' may be different. The dimension of the cone itself is of course invariant under linear isomorphisms.

We shall now take the matricial nature of the spectrahedral cones into account. Naturally, two spectrahedral cones should be considered isomorphic if one cone can be bijectively mapped to the other cone by a coordinate transformation. Such a transformation acts on the matrices in the source space by a map $X \mapsto AXA^T$, where A is a fixed invertible matrix. In order to accommodate different matrix sizes, we allow the matrices in the cone of smaller matrix size to be padded with zeros before the coordinate transformation. Equivalently, we may relax the invertibility condition on the coordinate transformation matrix A and allow it to be rectangular of full column rank. This leads to the following definition.

Definition 2.2. Let $K \subset \mathcal{S}_+^n$, $K' \subset \mathcal{S}_+^{n'}$ be spectrahedral cones, and suppose that $n \leq n'$. We call K, K' *isomorphic* if there exists an injective linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ with coefficient matrix $A \in \mathbb{R}^{n' \times n}$ of full column rank such that the induced injective linear map $\tilde{f} : \mathcal{S}^n \rightarrow \mathcal{S}^{n'}$ given by $\tilde{f} : X \mapsto AXA^T$ maps K bijectively onto K' .

It is not immediately evident that this notion is well-posed, i.e., defines an equivalence relation. For this we shall need the following reformulation of [11, Lemma 2.3].

Lemma 2.3. Let $K \subset \mathcal{S}_+^n$ be a spectrahedral cone, and let $m = \max_{X \in K} \text{rk } X$ be the maximal rank of the matrices in K . Then the set $R_{\max} = \{X \in K \mid \text{rk } X = m\}$ equals the (relative) interior of K , and the subspace $H(X) = \text{Im } X$ is constant over R_{\max} . \square

In other words, for every spectrahedral cone $K \subset \mathcal{S}_+^n$ there exists a coordinate system in \mathbb{R}^n such that the matrices in K are all of the form $X = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, where M is positive definite if and only if X is in the interior of K .

We shall now show that Definition 2.2 determines an equivalence relation.

- Reflexivity: By taking f to be the identity map we see that K is isomorphic to itself.
- Symmetry: If $n \neq n'$, then there is nothing to show. Let now $n = n'$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ with coefficient matrix $A \in \mathbb{R}^{n' \times n}$ realize an isomorphism between K and K' . Then the map f is bijective, and its inverse $f^{-1} : \mathbb{R}^{n'} \rightarrow \mathbb{R}^n$ with coefficient matrix $A^{-1} \in \mathbb{R}^{n \times n'}$ realizes an isomorphism between K' and K .
- Transitivity: Let $K \subset \mathcal{S}_+^n$, $K_1 \subset \mathcal{S}_+^{n_1}$, $K_2 \subset \mathcal{S}_+^{n_2}$ be spectrahedral cones and suppose that K is isomorphic to both K_1 and K_2 , the isomorphisms being generated by the injective maps f_1, f_2 , respectively. Assume that $n_1 \leq n_2$ without loss of generality. Then we have to construct an injective map $f : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ which generates an isomorphism between K_1 and K_2 . We have to distinguish several cases:

$n_1 \leq n \leq n_2$: Then $f_1 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^n$, $f_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n_2}$, and we may set $f = f_2 \circ f_1$.

$n \leq n_1 \leq n_2$: Then $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n_2}$. It is not hard to see that there exists an injective map $f : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ such that $f_2 = f \circ f_1$. Any such map generates an isomorphism between K_1 and K_2 .

$n_1 \leq n_2 \leq n$: This is the non-trivial case, since an injective map $f : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ satisfying $f_1 = f_2 \circ f$ may not exist. Let A_i be the coefficient matrix of the map f_i , $i = 1, 2$. Denote the generic elements of K, K_1, K_2 by X, X_1, X_2 , respectively, and set $m = \max_{X \in K} \text{rk } X$. Then we also have $m = \max_{X_i \in K_i} \text{rk } X_i$, $i = 1, 2$, because $\text{rk } X_i = \text{rk } A_i X_i A_i^T$. By Lemma 2.3 we may assume that all elements of the cones K, K_1, K_2 are of the block-diagonal form $\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, where the $m \times m$ matrix M is positive definite if and only if the element is in the interior of the respective cone. Let now $X \in K$ be an element of maximal rank m , and let $X_1 \in K_1$, $X_2 \in K_2$ be its preimages under the isomorphisms generated by f_1, f_2 , respectively. The matrices X_1, X_2 are also of rank m . Then we have

$$X_i = \begin{pmatrix} M_i & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} = A_i \begin{pmatrix} M_i & 0 \\ 0 & 0 \end{pmatrix} A_i^T, \quad i = 1, 2,$$

where M, M_1, M_2 are positive definite $m \times m$ matrices. Partition $A_i = \begin{pmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{pmatrix}$, where the block $A_{i,11}$ is of size $m \times m$. Then we obtain $A_{i,11}M_iA_{i,11}^T = M$, $A_{i,21}M_iA_{i,11}^T = 0$, $i = 1, 2$. Since M, M_i are invertible, it follows from the first equation that $A_{i,11}$ is invertible. Then the second equation yields $A_{i,21} = 0$. Define the invertible $m \times m$ matrix $A_{11} = A_{2,11}^{-1}A_{1,11}$, let A_{22} be an arbitrary $(n_2 - m) \times (n_1 - m)$ matrix of full column rank, and set $A = \text{diag}(A_{11}, A_{22})$. Then A is an $n_2 \times n_1$ matrix of full column rank, and we have by construction $AX_1A^T = X_2$ for all pairs $(X_1, X_2) \in K_1 \times K_2$ such that $A_1X_1A_1^T = A_2X_2A_2^T$. Hence the injective map f defined by the coefficient matrix A generates the sought isomorphism between K_1, K_2 .

Isomorphisms in the sense of Definition 2.2 are also linear isomorphisms, i.e., in the sense of Definition 2.1. For general spectrahedral cones, the former is, however, a much stronger condition than the latter. For instance, a linear isomorphism between spectrahedral cones in general does not preserve the rank of the matrices in the cone, while the map $X \mapsto AXA^T$ is rank-preserving if A is of full column rank. In particular, isomorphisms in the sense of Definition 2.2 preserve Property 1.1. Whether a particular spectrahedral cone is ROG thus depends only on the isomorphism class of this cone.

We may now reformulate Lemma 2.3 in terms of Definition 2.2.

Definition 2.4. We call a spectrahedral cone *non-degenerate* if its interior consists of positive definite matrices.

Lemma 2.5. Let K be a spectrahedral cone and set $m = \max_{X \in K} \text{rk } X$. Then there exists a non-degenerate spectrahedral cone $K' \subset \mathcal{S}_+^m$ which is isomorphic to K . Every non-degenerate spectrahedral cone which is isomorphic to K consists of matrices of size $m \times m$.

Proof. Set $H = \text{Im } X \subset \mathbb{R}^n$, where $X \in K$ is a matrix of maximal rank m . Identify H with the space \mathbb{R}^m by introducing an arbitrary coordinate system on H . Then the inclusion $i : H \rightarrow \mathbb{R}^n$ defines an injective map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$. By Lemma 2.3 the interior of K and hence the whole cone K lies in the image of the injective map $\tilde{f} : \mathcal{S}^m \rightarrow \mathcal{S}^n$ induced by f . The preimage $\tilde{f}^{-1}[K]$ is then a non-degenerate spectrahedral cone $K' \subset \mathcal{S}_+^m$. By construction K' is isomorphic to K by virtue of the map f . The last assertion of the lemma follows from the fact that the quantity $\max_{X \in K'} \text{rk } X$ is constant over the isomorphism class of K , and the matrix size is for a non-degenerate spectrahedral cone always equal to the maximal rank. \square

2.2 Notations

In this subsection we introduce some notations which simplify the expositions in the next sections.

For $n \in \mathbb{N}$, define two operators $\mathcal{L}_n, \mathcal{F}_n$ from the set of linear subspaces of \mathbb{R}^n into the set of linear subspaces of \mathcal{S}^n and the set of faces of the cone \mathcal{S}_+^n , respectively. Let $H \subset \mathbb{R}^n$ be a linear subspace. Then $\mathcal{L}_n(H), \mathcal{F}_n(H)$ will be defined as the linear span and the convex hull of the set $\{xx^T \in \mathcal{S}^n \mid x \in H\}$, respectively. Note that $\mathcal{F}_n(H)$ is isomorphic to the cone $\mathcal{S}_+^{\dim H}$. For a matrix $X \in \mathcal{S}_+^n$, the smallest face of \mathcal{S}_+^n containing X is then given by $\mathcal{F}_n(\text{Im } X)$.

In order to indicate the size n of the matrices making up a spectrahedral cone K , we shall write $K = L \cap \mathcal{S}_+^n$ or $K \subset \mathcal{S}_+^n$, where $L \subset \mathcal{S}^n$ is a linear subspace. Later in the paper we shall also work with ROG cones as abstract convex conic subsets of a real vector space. They may then have representations in matrix spaces of different sizes.

Let us also define an operator \mathcal{H}_n from the set of spectrahedral cones $K \subset \mathcal{S}_+^n$ to the set of linear subspaces of \mathbb{R}^n . For $X \in K$ a matrix of maximal rank, we set $\mathcal{H}_n(K) = \text{Im } X$. By Lemma 2.3 the image $\text{Im } X$ does not depend on the choice of X , and \mathcal{H}_n is indeed well-defined. The operator \mathcal{H}_n maps K to the whole space \mathbb{R}^n if and only if K is non-degenerate.

3 Basic properties of ROG cones

In Subsection 3.1 we shall consider ROG cones from the viewpoint of real algebraic geometry. This approach has been quite successful in the study of spectrahedral cones in general, by describing the boundary of these cones as subsets of the zero set of some hyperbolic polynomial. For a certain subclass of spectrahedral cones the minimal such polynomial is precisely the determinant of the matrices making

up the cone, and the ROG cones are shown to belong to this subclass. This links the rank of the matrices in the ROG cone to the degree of the cone as an algebraic set.

We then pass on to the facial structure of ROG cones in Subsection 3.2. We show that the facial hierarchy of ROG cones is much more tightly bound to the rank of the matrices in the faces than for general spectrahedral cones. The main result of this subsection is the equality of rank and Carathéodory number for the elements of ROG cones, a relation which is familiar and widely used for the full matrix cone \mathcal{S}_+^n .

Subsection 3.3 contains the main result of the paper. It states that if some convex cone is linearly isomorphic to some ROG cone, then all such ROG cones must be mutually isomorphic as spectrahedral cones. The non-triviality of this assertion comes from the fact that the isomorphism of convex cones as subsets of a real vector space is a much weaker notion than the isomorphism between spectrahedral cones, the latter taking into account also the structure of the matrices making up the cones. While the first two subsections in this section use only elementary tools, in Subsection 3.3 we will need to consider a certain property of the Plücker embedding of real Grassmanians. This result does not refer to spectrahedral cones and can be found in the Appendix.

3.1 Minimal defining polynomial

In this subsection we consider the boundary of spectrahedral and ROG cones as a subset of the zero locus of a polynomial. The main goal is to unveil the relation between these polynomials and the determinant of the matrices in the cone. The material in this subsection is basically an application of the theory of algebraic interiors which has been elaborated in [11].

Definition 3.1. [11, Section 2.2] A closed set $C \subset \mathbb{R}^m$ is an *algebraic interior* if there exists a polynomial p on \mathbb{R}^m such that C equals the closure of a connected component of the set $\{x \in \mathbb{R}^m \mid p(x) > 0\}$. Such a polynomial is called a *defining polynomial* of the algebraic interior.

Lemma 3.2. [11, Lemma 2.1] *Let C be an algebraic interior. Then the defining polynomial p of C with minimal degree is unique up to multiplication by a positive constant. Any other defining polynomial of C is divisible by p .*

Definition 3.3. The defining polynomial with minimal degree of an algebraic interior C is called *minimal defining polynomial*. The *degree* of C is defined as the degree of the minimal defining polynomial.

Lemma 3.4. [11, Theorem 2.2] *Every spectrahedron is a convex algebraic interior.*

From Lemma 3.2 it follows that the minimal defining polynomial of a spectrahedral cone is invariant under linear isomorphisms up to a multiplicative positive constant. It also follows that it is homogeneous. Indeed, under a homothety of the cone the minimal defining polynomial transforms to another minimal defining polynomial, which must differ from the original one by a multiplicative positive constant.

For a non-degenerate spectrahedral cone $K \subset \mathcal{S}_+^n$, a defining polynomial of K is given by the restriction of the determinant in \mathcal{S}^n to $\text{span } K$. We shall call this polynomial the *determinantal defining polynomial*. Since $\det(XA) = (\det A)^2 \cdot \det X$ for square matrices A, X , the determinantal defining polynomials of two isomorphic non-degenerate spectrahedral cones differ only by a multiplicative positive constant. We may hence extend the notion of determinantal defining polynomial to the whole isomorphism class of a non-degenerate spectrahedral cone, keeping in mind that for degenerate cones the polynomial is determined only up to a positive constant factor. The degree of the determinantal defining polynomial for general spectrahedral cones K is by virtue of Lemma 2.5 equal to $\max_{X \in K} \text{rk } X$.

In contrast to the minimal defining polynomial, the determinantal defining polynomial is in general not invariant under linear isomorphisms.

Example 3.5. Consider the three-dimensional Lorentz cone $L_3 = \{x = (x_0, x_1, x_2)^T \in \mathbb{R}^3 \mid x_0 \geq \sqrt{x_1^2 + x_2^2}\}$ and two of its spectrahedral representations $K_i = \{A_i(x) \mid x \in L_3\}$, $i = 1, 2$, given by

$$A_1(x) = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & 0 \\ x_2 & 0 & x_0 \end{pmatrix} \in \mathcal{S}^3, \quad A_2(x) = \begin{pmatrix} x_0 + x_1 & x_2 \\ x_2 & x_0 - x_1 \end{pmatrix} \in \mathcal{S}^2.$$

It is easily seen that $A_i(x) \succeq 0$ if and only if $x \in L_3$, so K_1, K_2 are indeed spectrahedral representations of L_3 . Clearly they are both non-degenerate. However, the corresponding determinantal defining

polynomials are given by $p_{1,\det} = x_0(x_0^2 - x_1^2 - x_2^2)$, $p_{2,\det} = x_0^2 - x_1^2 - x_2^2$ and are hence not proportional, while the minimal defining polynomial of L_3 is given by $p_{\min} = x_0^2 - x_1^2 - x_2^2$. This proves that the linearly isomorphic spectrahedral cones K_1, K_2 are not isomorphic in the sense of Definition 2.2.

We now come to the main result in this subsection, namely that for ROG cones the determinantal and minimal defining polynomials coincide. Actually, we shall prove this assertion for a somewhat larger subclass of spectrahedral cones².

Lemma 3.6. *Let $K = L \cap \mathcal{S}_+^n$ be a non-degenerate spectrahedral cone. Suppose that there exist linearly independent vectors $x_1, \dots, x_n \in \mathbb{R}^n$ such that $x_i x_i^T \in K$, $i = 1, \dots, n$. Then the determinantal defining polynomial d of K is a minimal defining polynomial.*

Proof. Denote the linear span of the matrices $x_1 x_1^T, \dots, x_n x_n^T \in K$ by D , and the intersection $D \cap K$ by K_D . We have $D \subset L$, and hence $D \cap \mathcal{S}_+^n = D \cap L \cap \mathcal{S}_+^n = K_D$. However, in the coordinates defined by the basis $\{x_1, \dots, x_n\}$ of \mathbb{R}^n the subspace $D \subset \mathcal{S}^n$ is the subspace of diagonal matrices. Hence $K_D = D \cap \mathcal{S}_+^n$ equals the convex conic hull of $\{x_1 x_1^T, \dots, x_n x_n^T\}$, which in turn is linearly isomorphic to the nonnegative orthant \mathbb{R}_+^n . Moreover, the relative interior of K_D consists of positive definite matrices and is hence contained in the relative interior of K . On the other hand, the boundary of K_D is contained in the boundary of K by Lemma 2.3.

Let $p : L \rightarrow \mathbb{R}$ be a minimal defining polynomial of K . Since the determinantal defining polynomial d has degree n , the degree of p is at most n . By Lemma 3.2 p divides d . Since $d > 0$ on the relative interior of K , we also have $p > 0$ on the relative interior of K . Hence $p > 0$ on the relative interior of K_D . On the other hand, $p = 0$ on the boundary of K_D , because $p = 0$ on the boundary of K . Therefore the restriction of p on D is a defining polynomial for the cone $K_D \cong \mathbb{R}_+^n$.

However, the degree of the algebraic interior \mathbb{R}_+^n is n , and hence p has degree at least n . It follows that $\deg p = n$, and d must be a minimal defining polynomial of K . \square

Theorem 3.7. *The determinantal defining polynomial of a ROG spectrahedral cone K is a minimal defining polynomial.*

Proof. Recall that both the determinantal and the minimal defining polynomial is invariant under isomorphisms up to multiplication by a positive constant. We may then assume without loss of generality that $K \subset \mathcal{S}_+^n$ is non-degenerate, otherwise we pass to an isomorphic non-degenerate ROG spectrahedral cone by virtue of Lemma 2.5.

Let $X \in K$ be positive definite. Since K is ROG, there exist vectors $x_1, \dots, x_N \in \mathbb{R}^n$ such that $X = \sum_{i=1}^N x_i x_i^T$ and $x_i x_i^T \in K$ for all $i = 1, \dots, N$. By virtue of $X \succ 0$ the linear span of $\{x_1, \dots, x_N\}$ equals \mathbb{R}^n . In particular, among the x_i there are n linearly independent vectors, let these be x_1, \dots, x_n . The proof is concluded by application of Lemma 3.6. \square

We may now link the degree of a ROG spectrahedral cone to the rank of the matrices in the cone.

Corollary 3.8. *The degree of a ROG spectrahedral cone K is given by $\deg K = \max_{X \in K} \text{rk } X$.*

Proof. The right-hand side of the relation is the degree of the determinantal defining polynomial of K , while the left-hand side is the degree of the minimal defining polynomial. The assertion now follows from Theorem 3.7. \square

In this subsection we have shown that for the subclass of ROG spectrahedral cones, two different kinds of associated polynomials coincide. These are on the one hand the *determinantal defining polynomial*, which is determined by the matricial structure of the elements of the cone and is invariant under isomorphisms in the sense of Definition 2.2, and on the other hand the *minimal defining polynomial*, which is a notion from real algebraic geometry and is invariant under linear isomorphisms in the sense of Definition 2.1.

²The extension from ROG cones to the subclass considered in Lemma 3.6 is due to Gregory Blekherman.

3.2 Facial structure and rank

In this subsection we study the facial structure of ROG cones and its connection to the Carathéodory number. This allows us to establish a number of representation lemmas which bound the number of rank 1 matrices which enter the sum in Property 1.1. The results in this subsection follow from properties of the facial structure of general spectrahedral cones and from standard convex analysis arguments.

We shall call an element of a cone K *extreme* if it generates an extreme ray of K .

Lemma 3.9. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone. Then the set of extreme elements of K is given by $\{X \in K \mid \text{rk } X = 1\}$.*

Proof. Since K is ROG, every $X \in K$ with $\text{rk } X > 1$ can be represented as sum of elements $X_i \in K$ of rank 1. Hence such X cannot be extreme. On the other hand, every $X \in K$ with $\text{rk } X = 1$ generates an extreme ray of \mathcal{S}_+^n . Extremality in K for such X follows immediately. \square

Let us recall the results of [18] on the facial structure of general spectrahedral cones. Let $K = L \cap \mathcal{S}_+^n$ be a spectrahedral cone. Then the faces of K are given by the intersections of L with the faces of \mathcal{S}_+^n [18, Theorem 1], see also [22, Prop. 2.1]. In particular, the kernel of the matrices $X \in K$ is constant over the relative interior of each face of K , and every face of K is exposed [18, Corollary 1]. It follows that the faces of spectrahedral cones are also spectrahedral cones.

The smallest face of $K = L \cap \mathcal{S}_+^n$ containing a matrix $X \in K$ is given by the intersections $L \cap \mathcal{F}_n(\text{Im } X) = L \cap \mathcal{S}_+^n \cap \mathcal{L}_n(\text{Im } X) = K \cap \mathcal{L}_n(\text{Im } X)$, because $\mathcal{F}_n(\text{Im } X)$ is the smallest face of \mathcal{S}_+^n containing X . The smallest face of \mathcal{S}_+^n containing K is given by $\mathcal{F}_n(\text{Im } X)$, where X is an arbitrary matrix in the interior of K . Here the operators $\mathcal{F}_n, \mathcal{L}_n$ are defined in Subsection 2.2.

Lemma 3.10. *Every face of a ROG cone is a ROG cone.*

Proof. Let $K = L \cap \mathcal{S}_+^n$ be a ROG cone and $K' \subset K$ a face of K . Then there exists a face F of \mathcal{S}_+^n such that $K' = L \cap F$, e.g., $F = (\mathcal{F}_n \circ \mathcal{H}_n)(K')$. Let $X \in K'$ be an arbitrary nonzero matrix. Since $X \in K$ and K is ROG, there exist rank 1 matrices $X_1, \dots, X_N \in K$ such that $X = \sum_{i=1}^N X_i$. At the same time, $X \in F$. Since F is a face of \mathcal{S}_+^n , the rank 1 matrices $X_i \in \mathcal{S}_+^n$ must also be elements of this face. It follows that $X_i \in K'$, and X can be represented as sum of rank 1 matrices in K' . Thus K' is ROG. \square

Definition 3.11. [8, p.59] Let $K \subset \mathbb{R}^m$ be a closed pointed convex cone. The *Carathéodory number* $\kappa(x)$ of a point $x \in K$ is the minimal number k such that there exist extreme elements x_1, \dots, x_k of K satisfying $x = \sum_{i=1}^k x_i$.

The *Carathéodory number* $\kappa(K)$ of the cone K is the maximum of $\kappa(x)$ over $x \in K$.

Lemma 3.12. *Let $K = L \cap \mathcal{S}_+^n$ be a spectrahedral cone. The Carathéodory number of $X \in K$ satisfies $\kappa(X) \leq \text{rk } X$.*

Proof. We proceed by induction. If $\text{rk } X \leq 1$, then by virtue of Lemma 3.9 we trivially have $\kappa(X) = \text{rk } X$. Suppose the relation $\kappa(X) \leq \text{rk } X$ is proven for $\text{rk } X \leq k-1$, and let $X \in K$ with $\text{rk } X = k \geq 2$.

Without loss of generality we may assume $n = k$, otherwise we replace K by $K_X = L \cap \mathcal{F}_n(\text{Im } X)$, the minimal face of K which contains X . Neither the rank nor the Carathéodory number of X will change by this substitution of the ambient cone, but now $\mathcal{F}_n(\text{Im } X) \cong \mathcal{S}_+^k$ and K_X can be seen as a spectrahedral cone defined by $k \times k$ matrices.

Then the boundary of K consists of matrices Y with $\text{rk } Y < k = n$, and hence $\kappa(Y) < k$ by the induction hypothesis. Let $E \in K$ an extreme element of K , normalized such that $\text{tr } E = \text{tr } X$. Consider the compact line segment l which is defined by the intersection of K with the affine line passing through X and E . Since X is in the interior of K , it is also in the interior of the segment l . One endpoint of l is given by E , while the other one is some matrix $Y \in \partial K$. Then there exists $\lambda \in (0, 1)$ such that $X = \lambda E + (1 - \lambda)Y$. Hence $\kappa(X) \leq \kappa(E) + \kappa(Y) \leq 1 + (k-1) = k$. This completes the proof. \square

Lemma 3.13. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone. The Carathéodory number of $X \in K$ is given by $\kappa(X) = \text{rk } X$.*

Proof. We have $\kappa(X) \geq \text{rk } X$, because by virtue of Lemma 3.9 all generators of extreme rays of K have rank 1, and a matrix X cannot be the sum of less than $\text{rk } X$ matrices of rank 1. On the other hand, $\kappa(X) \leq \text{rk } X$ by Lemma 3.12. \square

Corollary 3.14. *The Carathéodory number of a ROG cone equals its degree.*

Proof. The claim follows immediately from Lemma 3.13 and Corollary 3.8. \square

Corollary 3.15. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone, and let $X \in K$ be an element of rank k . Then there exist rank 1 matrices $X_i = x_i x_i^T \in K$, $i = 1, \dots, k$, such that $X = \sum_{i=1}^k X_i$ and the vectors x_1, \dots, x_k are linearly independent.*

Proof. The Corollary is a consequence of Lemmas 3.9 and 3.13. \square

Corollary 3.16. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone of degree $d = \deg K$. Then there exist d linearly independent vectors $r_1, \dots, r_d \in \mathbb{R}^n$ such that $r_i r_i^T \in K$ for $i = 1, \dots, d$.*

Proof. The claim follows from Corollaries 3.8 and 3.15. \square

As a consequence, we have the following stand-alone result on the diagonalization of matrices in a ROG cone.

Lemma 3.17. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone, and let $X \in K$ be an element of rank k . Then there exists a basis of \mathbb{R}^n such that in the corresponding coordinates we have $X = \text{diag}(1, \dots, 1, 0, \dots, 0)$, and all diagonal matrices of the form $\text{diag}(d_1, \dots, d_k, 0, \dots, 0)$, where $d_i \geq 0$, $i = 1, \dots, k$, are in K .*

Proof. By Corollary 3.15 there exist linearly independent vectors $x_1, \dots, x_k \in \mathbb{R}^n$ such that $X_i = x_i x_i^T \in K$, $i = 1, \dots, k$, and $X = \sum_{i=1}^k X_i$. Extend the set $\{x_1, \dots, x_k\}$ to a basis of \mathbb{R}^n , then in the coordinates defined by this basis we have $X = \text{diag}(1, \dots, 1, 0, \dots, 0)$.

Moreover, for all $d_1, \dots, d_k \geq 0$ we have $\sum_{i=1}^k d_i x_i x_i^T \in K$, and in the coordinates defined above this matrix has the form $\text{diag}(d_1, \dots, d_k, 0, \dots, 0)$. \square

In this subsection we have shown that the equality between the Carathéodory number of a matrix $X \in \mathcal{S}_+^n$ and its rank which is trivially valid for the cone \mathcal{S}_+^n extends to ROG spectrahedral cones in general. This allowed us to establish the representation result Corollary 3.15 and the existence result Corollary 3.16. Lemma 3.10 asserts that the subclass of ROG cones is closed under the operation of taking faces, a result which is known to be valid for general spectrahedral cones too.

3.3 Isomorphisms and linear isomorphisms

This subsection contains the main structural result of the paper, namely that the notion of linear isomorphism of convex cones from Definition 2.1 and the notion of isomorphism of spectrahedral cones from Definition 2.2 coincide on the class of ROG spectrahedral cones. More precisely, we show that if two ROG spectrahedral cones are linearly isomorphic, then they are also isomorphic as spectrahedral cones. The proof requires Lemma A.5, which follows from an auxiliary result on the image of the Plücker embedding of real Grassmanians. These are provided in the Appendix.

Theorem 3.18. *Let $K \subset \mathcal{S}_+^n, K' \subset \mathcal{S}_+^{n'}$ be linearly isomorphic ROG cones. Then they are also isomorphic in the sense of Definition 2.2.*

Proof. By Lemma 2.5 we may assume without loss of generality that the cones K, K' are non-degenerate. By Lemma 3.2 the degrees of K and K' coincide, and hence by Corollary 3.8 also the maximal ranks n, n' of matrices in the cones K, K' , respectively, coincide. We may thus consider both cones as linear sections of the matrix cone \mathcal{S}_+^n . Let $L, L' \subset \mathcal{S}_+^n$ be the linear hulls of K, K' , respectively, and set $m = \dim L = \dim L'$.

Let $\tilde{f} : L \rightarrow L'$ be a bijective linear map realizing the linear isomorphism between K and K' . Let $x_1, \dots, x_m \in \mathbb{R}^n$ be such that the set $\{x_i x_i^T \mid i = 1, \dots, m\}$ forms a basis of L . This is possible because K is a ROG cone. Note also that the vectors x_1, \dots, x_m span the whole space \mathbb{R}^n because L contains non-singular matrices. For every $i = 1, \dots, m$ we have that $x_i x_i^T$ is an extreme element of K by Lemma 3.9. Its image $\tilde{f}(x_i x_i^T)$ must then be an extreme element of K' , and again by Lemma 3.9 a positive semi-definite rank 1 matrix. Hence there exist nonzero vectors $y_i \in \mathbb{R}^n$ such that $\tilde{f}(x_i x_i^T) = y_i y_i^T$.

Moreover, the images $\tilde{f}(x_i x_i^T)$ form a basis of L' , because \tilde{f} is a bijection, and the vectors y_1, \dots, y_m span \mathbb{R}^n because L' contains non-singular matrices.

Denote the determinantal polynomial on \mathcal{S}^n by d . Then $p = d|_L$, $p' = d|_{L'}$ are the determinantal defining polynomials of K, K' , respectively. By Theorem 3.7 both p, p' are minimal defining polynomials. By Lemma 3.2 there exists a positive constant $c > 0$ such that $p = c \cdot (p' \circ \tilde{f})$.

Then the conditions of Lemma A.5 are fulfilled and by this lemma there exists an automorphism of \mathcal{S}^n , given by the map $X \mapsto AXA^T$ for some non-singular matrix A , which coincides with \tilde{f} on L and hence maps K bijectively onto K' . This completes the proof. \square

Theorem 3.18 states that the geometry of a ROG cone as a subset of real space determines its representations as linear sections of a positive semi-definite matrix cone uniquely up to isomorphisms in the sense of Definition 2.2. Of course, this does not preclude the existence of other, nonisomorphic, representations as a spectrahedral cone, but in these the cone will not be ROG. For instance, the spectrahedral cone K_2 in Example 3.5 is ROG, but the linearly isomorphic spectrahedral cone K_1 is not. In the sequel, when we speak of a representation of a ROG cone, we will always mean a spectrahedral representation where the cone is ROG.

4 Construction of new ROG cones from given ones

In this section we consider several ways to construct ROG spectrahedral cones of higher degree from given ones. By iterating these procedures, one may construct ROG cones of arbitrarily high complexity.

4.1 Direct sums

In this subsection we consider direct sums of ROG cones and introduce the notion of a simple ROG cone³, which is a cone that cannot be represented as a non-trivial direct sum. First we shall consider general spectrahedral cones and pinpoint the difficulties which are associated to the notion of direct sum. Then we show that for ROG cones, the situation looks much more favorable.

Recall that in Subsection 2.1 we considered two different notions of isomorphisms of spectrahedral cones. The notion of linear isomorphism, given in Definition 2.1, disregarded the matricial structure of the cone, while the second, stronger notion in Definition 2.2 took it into account. Similarly we may define the notion of direct sum of spectrahedral cones in different ways, first disregarding the matricial nature of the cones and then taking it into account.

Definition 4.1. Let $K \subset \mathbb{R}^n$, $K' \subset \mathbb{R}^{n'}$ be convex cones. Their *direct sum* $K \oplus K'$ is defined as the set $\{(x, x') \in \mathbb{R}^n \oplus \mathbb{R}^{n'} \mid x \in K, x' \in K'\}$.

Definition 4.2. Let $K \subset \mathcal{S}_+^n$, $K' \subset \mathcal{S}_+^{n'}$ be spectrahedral cones. Their *direct sum* $K \oplus K'$ is defined as the set $\{\text{diag}(X, X') \in \mathcal{S}_+^{n+n'} \mid X \in K, X' \in K'\}$.

Note that for the direct sum $K \oplus K'$ of spectrahedral cones $K \subset \mathcal{S}_+^n$, $K' \subset \mathcal{S}_+^{n'}$, the ambient vector space in Definition 4.1 is the product $\mathcal{S}^n \times \mathcal{S}^{n'}$, while in Definition 4.2 it is the matrix space $\mathcal{S}^{n+n'}$. However, the former can be naturally regarded as a subspace of the latter, namely the subspace of appropriately partitioned block-diagonal matrices. With this identification, both definitions obviously lead to the same result. The direct sum $K \oplus K'$ is also a spectrahedral cone, because a block-diagonal matrix is positive semi-definite if and only if all blocks are. All these considerations naturally extend to an arbitrary number of factors.

Let now a spectrahedral cone K be isomorphic to a direct sum $K_1 \oplus K_2$ of convex cones in the sense of Definition 4.1. Note that the factors K_1, K_2 are faces of K , and faces of spectrahedral cones are spectrahedral cones. Therefore the factors K_1, K_2 inherit from K the structure of spectrahedral cones, and we may consider their direct sum in the sense of Definition 4.2. By construction this direct sum is linearly isomorphic to the original cone K , but it turns out that it is not necessarily isomorphic to K in the sense of Definition 2.2.

³We propose to reserve the notion *irreducible* for ROG cones $K \subset \mathcal{S}_+^n$ such that the real projective variety defined by the set $\{x \in \mathbb{R}^n \mid xx^T \in K\}$ is irreducible.

Example 4.3. Consider the spectrahedral cones $K = \{A(x_1, x_2) \mid (x_1, x_2)^T \in \mathbb{R}_+^2\}$, $K_i = \{A_i(x_i) \mid x_i \geq 0\}$, $i = 1, 2$, given by

$$A(x_1, x_2) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 + x_2 & 0 \\ 0 & 0 & x_2 \end{pmatrix}, \quad A_1(x_1) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2(x_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2 \end{pmatrix}.$$

The cone K is linearly isomorphic to a direct sum of two copies of \mathbb{R}_+ , and the cones K_1, K_2 are its faces corresponding to the factors. However, their direct sum in the sense of Definition 4.2 is given by $K_1 \oplus K_2 = \{\text{diag}(x_1, x_1, 0, 0, x_2, x_2) \mid (x_1, x_2)^T \in \mathbb{R}_+^2\}$. We have $\max_{X \in K} \text{rk } X = 3$, $\max_{X \in K_1 \oplus K_2} \text{rk } X = 4$, and K cannot be isomorphic to $K_1 \oplus K_2$.

We shall show that ROG cones behave nicely in this respect, and such a situation cannot arrive. However, the main result of this subsection is much stronger. We show that a ROG cone consisting of block-diagonal matrices must be a direct sum whose decomposition into factors is given by the block partition. This is not true for general spectrahedral cones, as the example above demonstrates. The cones K_1, K_2 both consist of block-diagonal matrices, but neither of them decomposes into factors.

Lemma 4.4. *Let K_1, \dots, K_m be ROG cones of degrees n_1, \dots, n_m . Then their direct sum $K = \oplus_{k=1}^m K_k$ is also a ROG cone, of degree $n = \sum_{k=1}^m n_k$. The cone K is isomorphic to a block-diagonal non-degenerate ROG cone K' with block sizes n_1, \dots, n_m , such that block k defines a non-degenerate ROG cone K'_k which is isomorphic to K_k .*

Proof. Let $X_k \in K_k$ be arbitrary and let $X = \text{diag}(X_1, \dots, X_m) \in K$. Since the factor cones K_k are ROG, every X_k decomposes into a sum of rank 1 matrices $r_{k,j} \in K_k$, $j = 1, \dots, \eta_k$. For every such rank 1 matrix $r_{k,j}$, the matrix $R_{k,j} = \text{diag}(0, \dots, 0, r_{k,j}, 0, \dots, 0)$ is a rank 1 matrix in K , where the non-zero block is located at position k . Then $X = \sum_{k=1}^m \sum_{j=1}^{\eta_k} R_{k,j}$ is a rank 1 decomposition of X as required in Property 1.1. Since $X \in K$ was an arbitrary element, the cone K is ROG.

By Corollary 3.8 and Lemma 2.5, for every $k = 1, \dots, m$ there exists a non-degenerate ROG cone $K'_k \subset \mathcal{S}_+^{n_k}$ which is isomorphic to K_k . Their direct sum $K' = \oplus_{k=1}^m K'_k$ is isomorphic to K and has the required block structure. It is also non-degenerate, because a block-diagonal matrix with all blocks being positive definite is itself positive definite. Hence K has degree $n = \sum_{k=1}^m n_k$ by Corollary 3.8. \square

Corollary 4.5. *Let K_1, \dots, K_m be convex cones and $K = \oplus_{k=1}^m K_k$ their direct sum in the sense of Definition 4.1. Then K has a ROG spectrahedral representation if and only if all cones K_k have ROG spectrahedral representations.*

Proof. If the K_k have ROG representations, then their direct sum in the sense of Definition 4.2 is ROG by Lemma 4.4 and defines the required representation of K .

Let now K have a ROG representation. Each of the cones K_k is linearly isomorphic to a face of K , and this face defines a ROG representation of K_k by Lemma 3.10. \square

We shall now apply the powerful Theorem 3.18 to the situation in the preceding corollary. Since all ROG spectrahedral representations of K are isomorphic, they are in particular isomorphic to the simple block-diagonal one. Hence every such representation has itself a particularly simple structure.

Lemma 4.6. *Let the cone $K = \oplus_{k=1}^m K_k$ be a direct sum of lower-dimensional cones in the sense of Definition 4.1 and suppose that K possesses a ROG spectrahedral representation $K \subset \mathcal{S}_+^n$. Then there exists a direct sum decomposition $\mathbb{R}^n = \oplus_{k=1}^m H_k$ into subspaces of dimensions $\dim H_k \geq \deg K_k$ such that the intersection $F_k = \mathcal{L}_n(H_k) \cap K$ is linearly isomorphic to K_k for all $k = 1, \dots, m$, and $K = \sum_{k=1}^m F_k$. If $n = \deg K$, then $\dim H_k = \deg K_k$ for $k = 1, \dots, m$.*

Proof. By Corollary 4.5 each factor cone K_k is ROG, denote its degree by n_k .

By Lemma 4.4 we have $\deg K = \sum_{k=1}^m n_k$ and K possesses a block-diagonal representation as a linear section of $\mathcal{S}_+^{\deg K}$ with block sizes n_k , such that block k defines a representation of the factor cone K_k . By Theorem 3.18 the original representation of K as a linear section of \mathcal{S}_+^n is isomorphic to this block-diagonal representation. Let $f : \mathbb{R}^{\deg K} \rightarrow \mathbb{R}^n$ be the injective linear map from Definition 2.2 which defines the isomorphism, and denote by $H \subset \mathbb{R}^n$ the image of f . The map f then puts the direct sum decomposition of $\mathbb{R}^{\deg K}$ defined by the block structure of the block-diagonal representation in correspondence to some direct sum decomposition $H = \oplus_{k=1}^m H'_k$, where $\dim H'_k = \deg K_k$. Let

$\mathbb{R}^n = \oplus_{k=1}^m H_k$ be an arbitrary direct sum decomposition such that $H'_k \subset H_k$ for all $k = 1, \dots, m$. By construction this decomposition has the required properties.

If $n = \deg K$, then f is bijective, and $H_k = H'_k$ is the only possible choice for H_k . It follows that $\dim H_k = \deg K_k$ in this case. \square

We now come to the main result of this subsection. The key idea is that in a block-diagonal rank 1 matrix, only one block is non-zero. Hence a block-diagonal ROG cone must be a direct sum.

Lemma 4.7. *Let $K = L \cap \mathcal{S}_+^n$ be a ROG cone. Let $\mathbb{R}^n = H_1 \oplus \dots \oplus H_m$ be a direct sum decomposition of \mathbb{R}^n and suppose that $L \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$. Then K is the sum of the ROG cones $K_k = K \cap \mathcal{L}_n(H_k)$, $k = 1, \dots, m$, and is canonically isomorphic to their direct sum.*

Proof. First note that the cones K_k are faces of K and hence indeed ROG cones by Lemma 3.10. Moreover, the sum $\sum_{k=1}^m K_k$ is canonically isomorphic to the direct sum $\oplus_{k=1}^m K_k$, because we have $\dim(\sum_{k=1}^m \mathcal{L}_n(H_k)) = \sum_{k=1}^m \dim \mathcal{L}_n(H_k)$.

Clearly $\sum_{k=1}^m K_k \subset K$, because $K_k \subset K$ for all k and K is a convex cone.

Let now $X \in K$ be arbitrary. By Property 1.1 there exist rank 1 matrices $X_i \in K$, $i = 1, \dots, N$, such that $X = \sum_{i=1}^N X_i$. Now for every i we have $X_i \in L \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$. Since X_i is rank 1, there must exist $k_i \in \{1, \dots, m\}$ such that $X_i \in \mathcal{L}_n(H_{k_i})$. It follows that $X_i \in \mathcal{L}_n(H_{k_i}) \cap K = K_{k_i}$. Therefore $X \in \sum_{k=1}^m K_k$, and hence $K \subset \sum_{k=1}^m K_k$.

Thus we get $K = \sum_{k=1}^m K_k$, which completes the proof. \square

Definition 4.8. We call a ROG cone K *simple* if it is not isomorphic to a nontrivial direct sum of lower-dimensional cones.

By Corollary 4.5 it is irrelevant for this definition whether we suppose a direct sum decomposition in the sense of Definition 4.1 or in the sense of Definition 4.2 here. Note that the decomposition of a cone K into simple factor cones in the sense of Definition 4.1 is unique up to a permutation of the factors. The next result provides another criterion for simplicity.

Lemma 4.9. *A non-degenerate ROG cone $K \subset \mathcal{S}_+^n$ is simple if and only if there does not exist a nontrivial decomposition $\mathbb{R}^n = H_1 \oplus \dots \oplus H_m$ such that $\text{span } K \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$.*

Proof. If K is not simple, then Lemma 4.6 applies with a nontrivial direct sum decomposition of \mathbb{R}^n . The assertion of this lemma then implies $\text{span } K \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$.

On the other hand, if there exists a nontrivial decomposition $\mathbb{R}^n = H_1 \oplus \dots \oplus H_m$ such that $\text{span } K \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$, then by Lemma 4.7 K is isomorphic to the direct sum of the ROG cones $K_k = K \cap \mathcal{L}_n(H_k)$, $k = 1, \dots, m$. Since $K \subset \mathcal{S}_+^n$ is non-degenerate, we have $\deg K_k = \dim H_k > 0$ for all k , and the direct sum is nontrivial. \square

Lemma 4.10. *Let $K \subset \mathcal{S}_+^n$ be a non-degenerate ROG cone. Then there exists a unique (up to a permutation of factors) direct sum decomposition $\mathbb{R}^n = H_1 \oplus \dots \oplus H_m$ such that K is the sum of the faces $K_k = \mathcal{L}_n(H_k) \cap K$, and such that the factor cones K_k are simple.*

Proof. The claim of the lemma follows from Lemma 4.6, applied to the unique decomposition of K into simple factor cones in the sense of Definition 4.1, and the fact that the subspaces H_k are uniquely determined by the faces K_k representing the factor cones. \square

Thus there are two different criteria that allow to check whether a ROG cone K is not simple. On the one hand, one may consider the geometric decomposition of K into factor cones, disregarding the matricial structure. On the other hand, one has the algebraic criterion whether in a non-degenerate representation, K is contained in the sum of two complementary faces of the ambient matrix cone.

4.2 Full extensions

In this subsection we consider a method for constructing larger spectrahedral cones K from smaller cones K' , such that K is the preimage of K' under a specific linear projection. The interest in this procedure is motivated by the fact that it preserves Property 1.1, and hence allows to construct larger ROG cones (both in terms of degree and of dimension) from smaller ones. The asserted invariance of Property 1.1 is the main result of this subsection.

Before describing the construction formally, we shall provide a non-formal explanation and an example. Let $n' < n$ and consider the partition $\mathbb{R}^n = H \oplus E$, where H is spanned by the first n' and

E by the last $n - n'$ basis vectors, and the corresponding partition of the matrices $X \in \mathcal{S}^n$ into four blocks X_{ij} , $i, j = 1, 2$. Given a spectrahedral cone $K' \subset \mathcal{S}_+^{n'}$, we define a cone $K \subset \mathcal{S}_+^n$ as the set of all matrices $X \in \mathcal{S}_+^n$ such that the upper left $n' \times n'$ submatrix of X is an element of K' . The crucial observation is that the cone $K \subset \mathcal{S}_+^n$ is a spectrahedral cone, obtained by imposing linear conditions on the block X_{11} only.

Example 4.11. Let $n' = 2$, $n = 3$, and let K' be the direct sum $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or equivalently, the cone of diagonal positive semi-definite matrices. The cone K is then given by

$$K = \left\{ \begin{pmatrix} a_1 & 0 & a_3 \\ 0 & a_2 & a_4 \\ a_3 & a_4 & a_5 \end{pmatrix} \succeq 0 \mid a_1, \dots, a_5 \in \mathbb{R} \right\}.$$

By a permutation of rows and columns one obtains that K is isomorphic to the cone of positive semi-definite tri-diagonal matrices. Both cones K', K are ROG.

The procedure above relies on a very specific decomposition $\mathbb{R}^n = H \oplus E$, determined by the chosen basis of \mathbb{R}^n . It is not hard to see that the essential objects linking the cones K' and K are the projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ which truncates the last $n - n'$ elements of vectors $x \in \mathbb{R}^n$ and the induced projection $\tilde{\pi} = \pi \otimes \pi : \mathcal{S}^n \rightarrow \mathcal{S}^{n'}$ which assigns to a matrix X its subblock X_{11} . In a coordinate-free setting, we thus have to depart from an arbitrary projection. By an appropriate choice of coordinates in \mathbb{R}^n and $\mathbb{R}^{n'}$ we may, however, always achieve the block partition described above.

Let $n' < n$ be positive integers and consider a surjective linear map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$. The map π induces a surjective linear map $\tilde{\pi} = \pi \otimes \pi : \mathcal{S}^n \rightarrow \mathcal{S}^{n'}$, acting on rank 1 matrices by $\tilde{\pi} : xx^T \mapsto \pi(x)\pi(x)^T$. Let $E \subset \mathbb{R}^n$ be the kernel of π , then the kernel of $\tilde{\pi}$ is given by the linear subspace $L_E \subset \mathcal{S}^n$ spanned by all matrices of the form $xy^T + yx^T$, $x \in \mathbb{R}^n$, $y \in E$. For any subspace $H \subset \mathbb{R}^n$ which is complementary to E , there exists a unique right inverse μ_H of π such that $\text{Im } \mu_H = H$. The map μ_H is injective and generates an injective map $\tilde{\mu}_H = \mu_H \otimes \mu_H : \mathcal{S}^{n'} \rightarrow \mathcal{S}^n$. The map $\tilde{\mu}_H$ is the unique right inverse to $\tilde{\pi}$ with image $\mathcal{L}_n(H)$. The following result formalizes the assertions made about the cones K', K above.

Lemma 4.12. *Assume above notations, and let $K' \subset \mathcal{S}_+^{n'}$ be a spectrahedral cone. Then the intersection $K = \tilde{\pi}^{-1}[K'] \cap \mathcal{S}_+^n$ is also a spectrahedral cone, and $K' = \tilde{\pi}[K]$. Moreover, for any subspace $H \subset \mathbb{R}^n$ which is complementary to E , the map $\tilde{\mu}_H$ is an isomorphism between K' and the face $\mathcal{L}_n(H) \cap K$ of K .*

Proof. By definition of K we have $\tilde{\pi}[K] \subset K'$.

Let now H be a complementary subspace to E . By definition $\tilde{\mu}_H$ is an isomorphism between K' and its image $\tilde{\mu}_H[K'] = \tilde{\pi}^{-1}[K'] \cap \mathcal{L}_n(H)$. However, $\tilde{\mu}_H[K'] \subset \mathcal{S}_+^n$, and hence $\tilde{\mu}_H[K'] = \mathcal{S}_+^n \cap \tilde{\pi}^{-1}[K'] \cap \mathcal{L}_n(H) = K \cap \mathcal{L}_n(H)$.

Moreover, since $\tilde{\mu}_H$ is a right inverse of $\tilde{\pi}$, we have $\tilde{\pi}[\mathcal{L}_n(H) \cap K] = K'$ and hence $K' \subset \tilde{\pi}[K]$, which proves that $K' = \tilde{\pi}[K]$.

There exists a linear subspace $L' \subset \mathcal{S}^{n'}$ such that $K' = L' \cap \mathcal{S}_+^{n'}$. The preimage $L = \tilde{\pi}^{-1}[L']$ is then a subspace of \mathcal{S}^n . We claim that $K = L \cap \mathcal{S}_+^n$.

Since $K' \subset L'$, we have $\tilde{\pi}^{-1}[K'] \subset L$ and hence $K \subset L \cap \mathcal{S}_+^n$.

On the other hand, $\tilde{\pi}[L \cap \mathcal{S}_+^n] \subset \tilde{\pi}[L] \cap \tilde{\pi}[\mathcal{S}_+^n] = L' \cap \mathcal{S}_+^{n'} = K'$, and hence $L \cap \mathcal{S}_+^n \subset \tilde{\pi}^{-1}[K']$. This yields $L \cap \mathcal{S}_+^n \subset K$. Thus $K = L \cap \mathcal{S}_+^n$ is a spectrahedral cone. \square

Since any two linear surjections $\pi_1, \pi_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ are conjugated by an automorphism of the source space \mathbb{R}^n , the isomorphism class of K depends only on K' and n , but not on the concrete realization of the surjection π . This implies that the structure of the cone K is fully determined by the smaller cone K' and motivates the following definition.

Definition 4.13. Let $n' < n$ be integers and let $K' \subset \mathcal{S}_+^{n'}$, $K \subset \mathcal{S}_+^n$ be spectrahedral cones. We call K a *full extension* of K' if there exists a surjective linear map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ such that $K = \tilde{\pi}^{-1}[K'] \cap \mathcal{S}_+^n$, where $\tilde{\pi} = \pi \otimes \pi$.

We now consider when a given spectrahedral cone $K \subset \mathcal{S}_+^n$ is a full extension of some smaller cone K' . The following result gives a sufficient condition.

Lemma 4.14. *Let $K \subset \mathcal{S}_+^n$ be a spectrahedral cone and suppose there exist a subspace $E \subset \mathbb{R}^n$ of dimension $k > 0$ and a subspace $L \subset \mathcal{S}^n$ such that $K = L \cap \mathcal{S}_+^n$ and $L_E \subset L$. Then for every linear surjective map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ with kernel E , the conditions of Definition 4.13 are satisfied with $n' = n - k$, $K' = \tilde{\pi}[K]$.*

Proof. We have $\tilde{\pi}^{-1}[\tilde{\pi}[K]] = K + \ker \tilde{\pi} = (L \cap \mathcal{S}_+^n) + L_E$.

Since $L_E \subset L$, we get $(L \cap \mathcal{S}_+^n) + L_E \subset L$ and hence $((L \cap \mathcal{S}_+^n) + L_E) \cap \mathcal{S}_+^n \subset K$.

On the other hand, we trivially have $K = L \cap \mathcal{S}_+^n \subset ((L \cap \mathcal{S}_+^n) + L_E) \cap \mathcal{S}_+^n$.

Hence $K = \tilde{\pi}^{-1}[\tilde{\pi}[K]] \cap \mathcal{S}_+^n$, which is what we had to show. \square

The conditions in Lemma 4.14 are also necessary for K to be a full extension. In order to see this, one may choose E equal to the kernel of the projection π and L as in the proof of Lemma 4.12.

We now come to the main result of this subsection.

Theorem 4.15. *Let $n' < n$ and let the spectrahedral cone $K \subset \mathcal{S}_+^n$ be a full extension of the spectrahedral cone $K' \subset \mathcal{S}_+^{n'}$. Then K is ROG if and only if K' is ROG.*

Proof. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n'}$ be a surjective linear map satisfying the conditions of Definition 4.13. By an appropriate choice of the coordinates in \mathbb{R}^n and $\mathbb{R}^{n'}$ we may achieve that the projection π truncates the last $n - n'$ elements of vectors $x \in \mathbb{R}^n$, and the map $\tilde{\pi}$ takes a matrix $X \in \mathcal{S}^n$ to its upper left subblock X_{11} , as in the explanation at the beginning of this subsection.

Let K be ROG. By Lemma 4.12 K' is isomorphic to a face of K , and by Lemma 3.10 faces of ROG cones are also ROG. This proves that K' is ROG.

Suppose now that K' is a ROG cone. Let $X \in K$ be arbitrary. Then $X_{11} = \tilde{\pi}(X) \in K'$ by Lemma 4.12, and there exist nonzero vectors $v_1, \dots, v_N \in \mathbb{R}^{n'}$ such that $X_{11} = \sum_{i=1}^N v_i v_i^T$ and $v_i v_i^T \in K'$ for all i . Let V be the $n' \times N$ matrix formed of the column vectors v_i . The condition $X \succeq 0$ implies that the columns of the block X_{12} are in the image of $X_{11} = VV^T$. Therefore there exists a $k \times N$ matrix W such that $X_{12} = VW^T$. Let the columns of W be $w_1, \dots, w_N \in \mathbb{R}^k$. We then have the representation

$$X = \begin{pmatrix} V \\ W \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & X_{22} - WW^T \end{pmatrix} = \sum_{i=1}^N \begin{pmatrix} v_i \\ w_i \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & X_{22} - WW^T \end{pmatrix}.$$

Denote the rank 1 matrix $\begin{pmatrix} v_i \\ w_i \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}^T$ by U_i , $i = 1, \dots, N$. By construction $U_i \succeq 0$ and its upper left $n' \times n'$ submatrix $v_i v_i^T$ is an element of K' . Hence $U_i \in \tilde{\pi}^{-1}[v_i v_i^T] \cap \mathcal{S}_+^n \subset K$ for all i . The $k \times k$ matrix $X_{22} - WW^T$ is the Schur complement of X_{11} in X and is hence positive semi-definite. It can then be written as a sum $\sum_{j=1}^{N'} z_j z_j^T$ with $z_j \in \mathbb{R}^k$. The rank 1 matrices $Z_j = \begin{pmatrix} 0 & 0 \\ 0 & z_j z_j^T \end{pmatrix}$ are also in K , and hence $X = \sum_{i=1}^N U_i + \sum_{j=1}^{N'} Z_j$ is a sum of rank 1 matrices in K . This shows that K is also ROG and proves the other direction of the equivalence. \square

Given a ROG cone $K' \subset \mathcal{S}_+^{n'}$, Theorem 4.15 allows us to construct ROG cones K consisting of matrices of size n for any $n > n'$.

It is not hard to see that under the conditions of Definition 4.13, $\max_{X \in K} \text{rk } X = n - n' + \max_{X' \in K'} \text{rk } X'$. In particular, the cone $K \subset \mathcal{S}_+^n$ is non-degenerate if and only if $K' \subset \mathcal{S}_+^{n'}$ is non-degenerate. If K', K are ROG cones, then by Corollary 3.8 $\deg K = n - n' + \deg K'$. Note also that the full extension of a ROG cone is always simple.

4.3 Intertwinings

In this subsection we present a way to construct new ROG cones from pairs of given ROG cones of smaller degree. Let us start with an example.

Example 4.16. Consider the cones $K_1 = \text{Han}_+^3$, $K_2 = \mathcal{S}_+^2$. A composite cone $K \subset \mathcal{S}_+^4$ can be constructed from these two, consisting of positive semi-definite matrices of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & a_3 & a_4 & 0 \\ a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & a_6 & a_7 \end{pmatrix}, \quad a_1, \dots, a_7 \in \mathbb{R}.$$

One recognizes the cone K_1 in the upper left 3×3 subblock of K , and the cone K_2 in the lower right 2×2 subblock. The subblocks intersect in a smaller central subblock of size 1. Both K_1 and K_2 are canonically isomorphic to faces of K corresponding to the subblocks, and K is equal to the *sum* of these faces.

For generic instances of this construction, the upper left subblock defining K_1 , the lower right subblock defining K_2 , and the central subblock representing their intersection can be of any (compatible) sizes. If the central subblock has size zero, then K is the *direct* sum $K_1 \oplus K_2$. For non-trivial central subblocks the composite cone K is a projection of the direct sum $K_1 \oplus K_2$. The crucial condition that forces the sum $K_1 + K_2$ to be a spectrahedral cone is that the intersection $K_1 \cap K_2$, which naturally has non-zero elements only in the central subblock, is isomorphic to a full matrix cone and contains *every* positive semi-definite matrix which has non-zero elements only in the central subblock. In the example above this condition holds because the variable a_5 parameterizes the whole central subblock and does not appear anywhere else. In general, there will be many non-equivalent ways to combine two given cones K_1, K_2 . Our interest in this procedure is based on the fact that the composite cone is ROG if and only if the smaller cones K_1, K_2 are.

We shall now formally define how given spectrahedral cones K_1, K_2 can be composed to yield the cone K . We shall work in a coordinate-free setting, independent of a specific choice of coordinates, or equivalently, a specific block decomposition of the involved matrices. One should keep in mind, however, that by an appropriate coordinate change one can always achieve the block-structured situation described above. For ease of notation, we introduce the following definition.

Definition 4.17. Let $K \subset \mathcal{S}_+^n$ be a spectrahedral cone. We call a face F of K *full* if it is also a face of \mathcal{S}_+^n . The number $k = \max_{X \in F} \text{rk } X$ is called the *rank* of the face.

For $i = 1, 2$, let $K_i \subset \mathcal{S}_+^{n_i}$ be spectrahedral cones possessing full faces $F_i \subset K_i$ of rank k . Let $H_i = \mathcal{H}_{n_i}(F_i) \subset \mathbb{R}^{n_i}$ be the k -dimensional linear subspaces corresponding to these faces. Let $\iota_i : \mathbb{R}^k \rightarrow \mathbb{R}^{n_i}$ be injective linear maps such that $\text{Im } \iota_i = H_i$. Consider the k -dimensional subspace $N = \{(\iota_1(x), -\iota_2(x)) \mid x \in \mathbb{R}^k\}$ of the direct sum $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$. Set $n = n_1 + n_2 - k$ and identify \mathbb{R}^n with the quotient space $(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2})/N$. Let $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^n$ be the natural embeddings of the factors \mathbb{R}^{n_i} into $(\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2})/N$, i.e., $f_1 : x_1 \mapsto (x_1, 0) + N$, $f_2 : x_2 \mapsto (0, x_2) + N$. Then the f_i are injective linear maps such that $f_1 \circ \iota_1 = f_2 \circ \iota_2$. Let $\tilde{\iota}_i = \iota_i \otimes \iota_i : \mathcal{S}^k \rightarrow \mathcal{S}^{n_i}$ and $\tilde{f}_i = f_i \otimes f_i : \mathcal{S}^{n_i} \rightarrow \mathcal{S}^n$ be the injective maps induced by ι_i, f_i , respectively. Then we have also $\tilde{f}_1 \circ \tilde{\iota}_1 = \tilde{f}_2 \circ \tilde{\iota}_2$.

The construction in the preceding paragraph ensures that the images $\tilde{f}_1[K_1], \tilde{f}_2[K_2]$ are isomorphic to K_1, K_2 , respectively, and that they intersect in a full face of rank k , namely $\tilde{f}_1[F_1] = \tilde{f}_2[F_2]$.

Definition 4.18. Assume above conditions. We call the cone $K = \tilde{f}_1[K_1] + \tilde{f}_2[K_2] \subset \mathcal{S}_+^n$ an *inter-twining* of the cones K_1, K_2 along the full faces F_i .

Remark 4.19. The cone K can be seen as the projection of the direct sum $K_1 \oplus K_2$ along the linear subspace generated by the set $\{(\tilde{\iota}_1(xx^T), -\tilde{\iota}_2(xx^T)) \mid x \in \mathbb{R}^k\} \subset \mathcal{S}^{n_1} \times \mathcal{S}^{n_2}$.

We have to show that K is indeed a spectrahedral cone. The following observation is crucial for the proof.

Lemma 4.20. Let $M = \begin{pmatrix} A & B & 0 \\ B^T & C & D \\ 0 & D^T & E \end{pmatrix}$ be a block-partitioned positive semi-definite matrix. Then

there exists a decomposition $C = C_1 + C_2$ such that the matrices $\begin{pmatrix} A & B \\ B^T & C_1 \end{pmatrix}, \begin{pmatrix} C_2 & D \\ D^T & E \end{pmatrix}$ are positive semi-definite.

Proof. The Schur complement of A in M is given by $\begin{pmatrix} C - B^T A^\dagger B & D \\ D^T & E \end{pmatrix}$ and is positive semi-definite.

Here A^\dagger is the pseudo-inverse of A , which is also positive semi-definite. Setting $C_1 = B^T A^\dagger B$, $C_2 = C - B^T A^\dagger B$ yields the desired decomposition. \square

Lemma 4.21. Assume the conditions of Definition 4.18 and let $L_i \subset \mathcal{S}^n$ be the linear hull of the image $\tilde{f}_i[K_i]$, $i = 1, 2$. Then $K = L \cap \mathcal{S}_+^n$, where $L = L_1 + L_2$. Moreover, we have $\tilde{f}_i[K_i] = \Lambda_i \cap K$, where $\Lambda_i = \text{Im } \tilde{f}_i = \mathcal{L}_n(\text{Im } f_i)$, and the cones $\tilde{f}_i[K_i]$ are faces of K , $i = 1, 2$.

Proof. Set $H = \text{Im } f_1 \cap \text{Im } f_2$. Then $\Lambda_1 \cap \Lambda_2 = \mathcal{L}_n(H) = L_1 \cap L_2$. By definition we have $L_i \subset \Lambda_i$ and $K = (L_1 \cap \mathcal{S}_+^n) + (L_2 \cap \mathcal{S}_+^n)$.

Introduce a direct sum decomposition $\mathbb{R}^n = H'_1 \oplus H \oplus H'_2$ such that $\text{Im } f_1 = H'_1 \oplus H$, $\text{Im } f_2 = H \oplus H'_2$. Adopt a coordinate system in \mathbb{R}^n which is adapted to this decomposition and partition the matrices in \mathcal{S}^n accordingly. Then every matrix in $\Lambda_1 + \Lambda_2$, and hence also in L , has the form $X = \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{12}^T & X_{22} & X_{23} \\ 0 & X_{23}^T & X_{33} \end{pmatrix}$. Moreover, every matrix whose only nonzero block is X_{22} is in $\Lambda_1 \cap \Lambda_2$ and hence in L .

Clearly $K = (L_1 \cap \mathcal{S}_+^n) + (L_2 \cap \mathcal{S}_+^n) \subset (L_1 + L_2) \cap \mathcal{S}_+^n = L \cap \mathcal{S}_+^n$. Let us show the reverse inclusion.

Let $X \in L \cap \mathcal{S}_+^n$ be an arbitrary matrix, partitioned as above. By Lemma 4.20 there exists a decomposition $X_{22} = X_{22,1} + X_{22,2}$ such that the matrices

$$X_1 = \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{12}^T & X_{22,1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \Lambda_1, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_{22,2} & X_{23} \\ 0 & X_{23}^T & X_{33} \end{pmatrix} \in \Lambda_2$$

are positive semi-definite. On the other hand, by virtue of $X \in L_1 + L_2$ there exists a decomposition $X = X_3 + X_4$ such that

$$X_3 = \begin{pmatrix} X_{11} & X_{12} & 0 \\ X_{12}^T & X_{22,3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in L_1, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_{22,4} & X_{23} \\ 0 & X_{23}^T & X_{33} \end{pmatrix} \in L_2.$$

We have $D_1 = X_1 - X_3 \in \Lambda_1 \cap \Lambda_2 \subset L_1$, $D_2 = X_2 - X_4 \in \Lambda_1 \cap \Lambda_2 \subset L_2$. Hence $X_1 = D_1 + X_3 \in L_1$, $X_2 = D_2 + X_4 \in L_2$. It follows that $X_1 \in \tilde{f}_1[K_1]$, $X_2 \in \tilde{f}_2[K_2]$. Therefore $X = X_1 + X_2 \in K$. Thus $L \cap \mathcal{S}_+^n \subset K$, which proves the first assertion of the lemma.

By construction we have $\tilde{f}_1[K_1] \subset \Lambda_1 \cap K$. Let us show the reverse inclusion.

Let $X \in \Lambda_1 \cap K$ be an arbitrary element. Since $K = \tilde{f}_1[K_1] + \tilde{f}_2[K_2]$, there exists a decomposition $X = X_1 + X_2$ such that $X_i \in \tilde{f}_i[K_i]$, $i = 1, 2$. We have $X, X_1 \in \Lambda_1$, and hence $X_2 \in \Lambda_1 \cap \Lambda_2 \cap \mathcal{S}_+^n = \mathcal{F}_n(H) \subset \tilde{f}_1[K_1]$. Hence $X \in \tilde{f}_1[K_1]$, which proves $\Lambda_1 \cap K \subset \tilde{f}_1[K_1]$.

The equality $\Lambda_2 \cap K = \tilde{f}_2[K_2]$ is shown in a similar way. \square

The isomorphism class of K depends not only on the cones K_i and the full faces $F_i \subset K_i$, but also on the maps ι_i , or more precisely, on the linear bijection defined between H_1 and H_2 by the map $\iota_2 \circ \iota_1^{-1}$, where ι_1^{-1} is an arbitrary left inverse of ι_1 , because it is this bijection which determines the subspace N .

Remark 4.22. The intertwining operation is not "associative" in the sense that the intertwining K_{123} of a cone K_3 with the intertwining K_{12} of cones K_1, K_2 can always be represented as an intertwining of K_2 with an intertwining K_{13} of K_1, K_3 . However, since every full face of K_{12} is a subset either of the face of K_{12} isomorphic to K_1 or of the face of K_{12} isomorphic to K_2 , there exists a permutation σ of the index set $\{1, 2\}$ such that K_{123} is an intertwining of $K_{\sigma(2)}$ with an intertwining $K_{\sigma(1)3}$ of the cones $K_{\sigma(1)}, K_3$.

We now come to the connection with ROG cones.

Lemma 4.23. *Assume the conditions of Definition 4.18. Then K is a ROG cone if and only if K_1, K_2 are ROG cones.*

Proof. If K is ROG, then K_1, K_2 are also ROG by Lemmas 3.10 and 4.21.

Assume that K_1, K_2 are ROG, then $\tilde{f}_1[K_1], \tilde{f}_2[K_2]$ are also ROG. Let $X \in K$ be arbitrary. Since $K = \tilde{f}_1[K_1] + \tilde{f}_2[K_2]$, there exist $X_i \in \tilde{f}_i[K_i]$, $i = 1, 2$, such that $X = X_1 + X_2$. Since $\tilde{f}_i[K_i]$ are ROG, both X_1 and X_2 can be represented as a sum of rank 1 matrices in $\tilde{f}_1[K_1]$ and $\tilde{f}_2[K_2]$, respectively. Hence X can be represented as a sum of rank 1 matrices in $\tilde{f}_1[K_1] \cup \tilde{f}_2[K_2] \subset K$. Thus K is ROG. \square

Remark 4.24. The preceding proof can in an obvious way be modified to show that if a spectrahedral cone is the sum of a finite number of its faces, then it is ROG if and only if all these faces are ROG.

Finally we shall consider the special case $k = 1$. This case is simpler than the general case in two respects. Firstly, a face F of a spectrahedral cone satisfying $\max_{X \in F} \text{rk } X = 1$ is always full. For a

ROG cone, the set of such faces equals the set of extreme rays. In particular, every ROG cone possesses full faces of rank 1. Therefore every two ROG cones can be intertwined along full faces of rank 1.

The second point is that given spectrahedral cones K_1, K_2 with full faces $F_i \subset K_i$ of rank 1, $i = 1, 2$, the isomorphism class of the intertwining of K_1 and K_2 along the faces F_1 and F_2 is even independent of the maps ι_1, ι_2 . This is because any two bijective maps $\iota, \iota' : \mathbb{R} \rightarrow \mathbb{R}$ can be conjugated by a homothety of the target space, and this homothety can be compensated for by a homothety of one of the cones K_1, K_2 . The cone constructed in Example 4.16 is, e.g., the only cone which can be constructed by an intertwining of Han_+^3 and \mathcal{S}_+^2 up to isomorphism, because every extreme ray of each of these cones can be taken to any other by an automorphism of the corresponding cone.

5 Examples of ROG cones

In this section we consider two nontrivial families of ROG cones. We show that the class of ROG cones defined by chordal graphs can be constructed from the full matrix cones \mathcal{S}_+^n by applying the constructive procedures presented in the previous section. We also provide an example of a continuous family of isomorphism classes of ROG cones.

5.1 Cones defined by chordal graphs

In this subsection we consider spectrahedral cones $K_G = L_G \cap \mathcal{S}_+^n$ defined by linear subspaces of the form $L_G = \{X \in \mathcal{S}^n \mid X_{ij} = 0 \ \forall (i, j) \notin E(G)\}$, where $E(G)$ is the edge set of a graph G on the vertices $1, \dots, n$. Note that the identity matrix is an element of K_G . Hence K_G has a nonempty intersection with the interior of \mathcal{S}_+^n , and the linear span of K_G equals L_G .

Lemma 5.1. *[1, Theorem 2.3], [17, Theorem 2.4] Assume above notations. Then the cone K_G is ROG if and only if the graph G is chordal.*

Chordal graphs are characterized by the condition that they admit a *perfect elimination ordering* of the vertices $1, \dots, n$. This is an ordering such that for every $k = 1, \dots, n$, the subset $N_k = \{l < k \mid (l, k) \in E(G)\} \cup \{k\}$ of vertices forms a *clique*, i.e., the subgraph of G defined by N_k is complete.

Lemma 5.2. *Let G be a chordal graph with vertex set $\{1, \dots, n\}$, and let K_G be the corresponding ROG cone. Then K_G can be constructed out of full matrix cones by iterated intertwinings or taking direct sums.*

Proof. Assume that the vertices are arranged in a perfect elimination ordering. For a subset $I \subset \{1, \dots, n\}$ of indices, define the linear subspace $H_I = \{x \in \mathbb{R}^n \mid x_i = 0 \ \forall i \notin I\}$. For $k = 1, \dots, n$, set $K_k = K_G \cap \mathcal{F}_n(H_{\{1, \dots, k\}})$.

Note that K_1 is isomorphic to the full matrix cone \mathcal{S}_+^1 . We shall now show for all $k = 2, \dots, n$ that the cone K_k is either an intertwining of K_{k-1} with a full matrix cone, or a direct sum $K_{k-1} \oplus \mathcal{S}_+^1$.

Since G is chordal, the set $N_k = \{l < k \mid (l, k) \in E(G)\} \cup \{k\}$ and its subset $N'_k = \{l < k \mid (l, k) \in E(G)\}$ define cliques of G . Therefore the faces $\mathcal{F}_n(H_{N_k}), \mathcal{F}_n(H_{N'_k})$ of \mathcal{S}_+^n are contained in K and are full faces of this cone. In particular, $\mathcal{F}_n(H_{N'_k})$ is a full face of both $\mathcal{F}_n(H_{N_k})$ and K_{k-1} . On the other hand, $K_k = K_{k-1} + \mathcal{F}_n(H_{N_k})$ by definition of N_k . Hence K_k is an intertwining of K_{k-1} with the full matrix cone $\mathcal{F}_n(H_{N_k})$ in case that $N'_k \neq \emptyset$, and a direct sum $K_{k-1} \oplus \mathcal{F}_n(H_{\{k\}})$ in case that $N'_k = \emptyset$.

The proof is completed by the observation that $K_G = K_n$. \square

Lemma 5.3. *Let G be a chordal graph with vertex set $\{1, \dots, n\}$, and let K_G be the corresponding ROG cone. Then $\deg K_G = n$, and K_G is simple if and only if G is connected.*

Proof. By construction $K_G \subset \mathcal{S}_+^n$ contains the identity matrix, and hence $\deg K_G = n$ by Corollary 3.8.

Suppose that K_G is not simple. Then there exists a nontrivial direct sum decomposition $\mathbb{R}^n = H \oplus H'$ such that for every rank 1 matrix $xx^T \in K_G$, either $x \in H$ or $x \in H'$. In particular, if $x = e_i$ is a canonical basis vector, then $e_i e_i^T \in K_G$ by construction of K_G and hence $e_i \in H \cup H'$ for all $i = 1, \dots, n$. Define the index sets $I = \{i \mid e_i \in H\}$ and $I' = \{i \mid e_i \in H'\}$. Then $I \cap I' = \emptyset$ and $I \cup I' = \{1, \dots, n\}$, because $\mathbb{R}^n = H \oplus H'$ is a direct sum decomposition. It follows that $H = \text{span}\{e_i \mid i \in I\}$ and $H' = \text{span}\{e_i \mid i \in I'\}$. Let now $x \in \mathbb{R}^n$ be a nonzero vector such that $X = xx^T \in K_G$. Then for every index pair $(i, j) \in I \times I'$ we have $x_i x_j = 0$ and hence $X_{ij} = 0$. From the fact that K_G is a ROG cone

it follows that $X_{ij} = 0$ for all $X \in \text{span } K_G = L_G$ in general for $(i, j) \in I \times I'$. But then $(i, j) \notin E(G)$, and there is no edge in G which connects the vertex subsets I, I' . Hence G is not connected.

Suppose, on the other hand, that G is not connected. Let I, I' be disjoint nonempty vertex sets such that $I \cup I' = \{1, \dots, n\}$ and there is no edge in G which connects I to I' . Then by definition for every $X \in L_G$ we have $X_{ij} = x_i x_j = 0$ for every index pair $(i, j) \in I \times I'$. Define subspaces $H = \text{span}\{e_i \mid i \in I\}$, $H' = \text{span}\{e_i \mid i \in I'\}$ of \mathbb{R}^n . Then $\mathbb{R}^n = H \oplus H'$ is by construction a nontrivial direct sum decomposition. It then follows that $L_G \subset \mathcal{L}_n(H) + \mathcal{L}_n(H')$, and the cone K_G is not simple. \square

5.2 A continuous family of non-isomorphic cones

In this subsection we construct a family of ROG cones in \mathcal{S}^6 whose isomorphism class depends on a real parameter. First we shall explain the construction informally. We begin with a full matrix cone \mathcal{S}_+^2 and intertwine consecutively four other copies of \mathcal{S}_+^2 with it along faces of rank 1. This singles out a quadruple of rank 1 faces in the original copy of \mathcal{S}_+^2 , or equivalently, a quadruple of points on the projective line \mathbb{RP}^1 . Now two composite cones of this form are isomorphic if and only if the corresponding quadruples of faces can be taken to each other by an automorphism of \mathcal{S}_+^2 , or equivalently, if the quadruples of points on \mathbb{RP}^1 are projectively equivalent. However, quadruples of points in \mathbb{RP}^1 possess a real invariant, the cross-ratio, which then parameterizes the isomorphism class of the composite cones.

We now define the cones formally. Fix mutually distinct angles $\varphi_1, \dots, \varphi_4 \in [0, \pi)$. For $\varphi \in [0, \pi)$, let $l(\varphi) \subset \mathbb{R}^2$ be the line through the origin with incidence angle φ . Then the lines $l(\varphi_1), \dots, l(\varphi_4)$ define a quadruple of points in real projective space \mathbb{RP}^1 .

Consider the 11-dimensional subspace $L_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} \subset \mathcal{S}^6$ of matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \cos \varphi_1 & \alpha_4 \cos \varphi_2 & \alpha_5 \cos \varphi_3 & \alpha_6 \cos \varphi_4 \\ \alpha_2 & \alpha_7 & \alpha_3 \sin \varphi_1 & \alpha_4 \sin \varphi_2 & \alpha_5 \sin \varphi_3 & \alpha_6 \sin \varphi_4 \\ \alpha_3 \cos \varphi_1 & \alpha_3 \sin \varphi_1 & \alpha_8 & 0 & 0 & 0 \\ \alpha_4 \cos \varphi_2 & \alpha_4 \sin \varphi_2 & 0 & \alpha_9 & 0 & 0 \\ \alpha_5 \cos \varphi_3 & \alpha_5 \sin \varphi_3 & 0 & 0 & \alpha_{10} & 0 \\ \alpha_6 \cos \varphi_4 & \alpha_6 \sin \varphi_4 & 0 & 0 & 0 & \alpha_{11} \end{pmatrix}, \quad \alpha_1, \dots, \alpha_{11} \in \mathbb{R}. \quad (3)$$

In the main result of this subsection, Lemma 5.5 below, we shall show that the 4-dimensional family of spectrahedral cones $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} = L_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} \cap \mathcal{S}_+^6$ is ROG and under the isomorphism equivalence relation projects to a 1-dimensional family of isomorphism classes. However, first we construct a sequence of intermediate cones, with $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$ as the last element, and show that each one is obtained by an intertwining of the preceding one with the full matrix cone \mathcal{S}_+^2 .

Let $H_0, \dots, H_4 \subset \mathbb{R}^6$ be the two-dimensional subspaces spanned by the columns of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \cos \varphi_1 & 0 \\ \sin \varphi_1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \cos \varphi_2 & 0 \\ \sin \varphi_2 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \cos \varphi_3 & 0 \\ \sin \varphi_3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \cos \varphi_4 & 0 \\ \sin \varphi_4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

respectively. Define subspaces $L_j = \sum_{i=0}^j \mathcal{L}_6(H_i) \subset \mathcal{S}^6$ and spectrahedral cones $K_j = L_j \cap \mathcal{S}_+^6$, $j = 0, \dots, 4$. Then $L_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} = L_4$, $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4} = K_4$.

Lemma 5.4. *The cone K_j , $j = 1, \dots, 4$, is a ROG cone given by the sum $\sum_{i=0}^j \mathcal{F}_6(H_i)$.*

Proof. We prove the lemma by induction over j . For $j = 0$ the assertion holds by construction.

Let now $j > 0$ and assume that $K_{j-1} = \sum_{i=0}^{j-1} \mathcal{F}_6(H_i)$ is a ROG cone. We have to show that $K_j = K_{j-1} + \mathcal{F}_6(H_j)$ is ROG. To this end we construct K_j as an appropriate intertwining of K_{j-1} and $\mathcal{S}_+^2 \simeq \mathcal{F}_6(H_j)$.

Note that the non-zero elements of the matrices in K_j are located in the upper left $(j+1) \times (j+1)$ subblock. For the sake of simplicity, we shall consider K_{j-1} as a subset of \mathcal{S}_+^j and K_j as a subset of \mathcal{S}_+^{j+1} . Set $k = 1$, $n_1 = j$, $n_2 = 2$, $n = j+1$. Define the injections $\iota_1 : \mathbb{R} \rightarrow \mathbb{R}^j$, $\iota_2 : \mathbb{R} \rightarrow \mathbb{R}^2$, $f_1 : \mathbb{R}^j \rightarrow \mathbb{R}^{j+1}$, $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^{j+1}$ by $\iota_1(1) = (\cos \varphi_j, \sin \varphi_j, 0, \dots, 0)^T$, $\iota_2(1) = (1, 0)^T$, $f_1(x_1, \dots, x_j) = (x_1, \dots, x_j, 0)^T$,

$f_2(x_1, x_2) = (x_1 \cos \varphi_j, x_1 \sin \varphi_j, 0, \dots, 0, x_2)$. By construction, the subspace L_j is the linear hull of the intertwining of K_{j-1} and \mathcal{S}_+^2 defined by these maps, and $\mathcal{F}_6(H_j)$ is the image of \mathcal{S}_+^2 under the induced map \tilde{f}_2 . Hence $K_j = K_{j-1} + \mathcal{F}_6(H_j)$ by Lemma 4.21 and K_j is ROG by Lemma 4.23. This completes the proof. \square

Lemma 5.5. *The cone $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$ is a ROG cone. Two cones $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$, $K_{\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4}$ are isomorphic if and only if the corresponding quadruples of lines $l(\varphi_1), \dots, l(\varphi_4) \subset \mathbb{R}^2$ and $l(\varphi'_1), \dots, l(\varphi'_4) \subset \mathbb{R}^2$ define projectively equivalent quadruples of points in \mathbb{RP}^1 .*

Proof. The first assertion of the lemma follows from Lemma 5.4 for $j = 4$.

Let us prove the second one. Consider cones $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$, $K_{\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4}$ for quadruples $(\varphi_1, \dots, \varphi_4)$, $(\varphi'_1, \dots, \varphi'_4)$ of mutually distinct angles. Let H_0, \dots, H_4 and H'_0, \dots, H'_4 , respectively, be the corresponding 2-dimensional subspaces of \mathbb{R}^6 as defined by the column spaces of the matrices (4). Note that $H_0 = H'_0$. By Lemma 5.4 the set $\{x \in \mathbb{R}^6 \mid xx^T \in K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}\}$ is given by the union $\bigcup_{j=0}^4 H_j$, and the set $\{x \in \mathbb{R}^6 \mid xx^T \in K_{\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4}\}$ by the union $\bigcup_{j=0}^4 H'_j$. The cones $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$ and $K_{\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4}$ are then isomorphic if and only if there exists an invertible linear map $f : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ which takes $\bigcup_{j=0}^4 H_j$ to $\bigcup_{j=0}^4 H'_j$.

Suppose that such a map f exists. The intersections $l(\varphi_i) = H_0 \cap H_i$, $l(\varphi'_i) = H'_0 \cap H'_i$, $i = 1, 2, 3, 4$, are 1-dimensional, while the intersections $H_i \cap H_j$, $H'_i \cap H'_j$, $i \neq j$, $i, j = 1, \dots, 4$, are 0-dimensional. Hence we must have $f[H_0] = H'_0$ and $f[H_i] = H'_{\sigma(i)}$, $i = 1, \dots, 4$, where $\sigma \in S_4$ is a permutation of the index set $\{1, \dots, 4\}$. Moreover, $f|_{H_0}[l_i] = l'_{\sigma(i)}$, $i = 1, \dots, 4$. It follows that $l(\varphi_1), \dots, l(\varphi_4) \subset H_0$ and $l(\varphi'_1), \dots, l(\varphi'_4) \subset H_0$ define projectively equivalent quadruples of points in the projectivization of H_0 .

Suppose now that the lines $l(\varphi_1), \dots, l(\varphi_4) \subset H_0$ and $l(\varphi'_1), \dots, l(\varphi'_4) \subset H_0$ define projectively equivalent quadruples of points in the projectivization of H_0 . Then there exists an invertible linear map $h : H_0 \rightarrow H_0$ and a permutation $\sigma \in S_4$ such that $h[l_i] = l'_{\sigma(i)}$, $i = 1, \dots, 4$. Let now $x_i \in H_i \setminus l_i$, $x'_i \in H'_i \setminus l'_i$, $i = 1, \dots, 4$, be arbitrary points. We then have $H_i = \text{span}(l_i \cup \{x_i\})$, $H'_i = \text{span}(l'_i \cup \{x'_i\})$, $i = 1, \dots, 4$. Moreover, $\text{span}(H_0 \cup \{x_1, x_2, x_3, x_4\}) = \text{span}(H_0 \cup \{x'_1, x'_2, x'_3, x'_4\}) = \mathbb{R}^6$. We then can extend the map h to a linear map $f : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ such that $f(x_i) = x'_{\sigma(i)}$, $i = 1, \dots, 4$. This map is invertible by construction and $f[H_i] = H'_{\sigma(i)}$, $i = 1, \dots, 4$. It follows that $f[\bigcup_{j=0}^4 H_j] = \bigcup_{j=0}^4 H'_j$, which completes the proof. \square

It is well-known that there exist infinitely many projectively non-equivalent quadruples of points in \mathbb{RP}^1 . The equivalence classes are parameterized by the orbits of the *cross-ratio* $\lambda(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (l_1, l_2; l_3, l_4) = \frac{(\cot \varphi_1 - \cot \varphi_3)(\cot \varphi_2 - \cot \varphi_4)}{(\cot \varphi_2 - \cot \varphi_3)(\cot \varphi_1 - \cot \varphi_4)}$ with respect to the action of the symmetric group S_4 on the arguments $\varphi_1, \dots, \varphi_4$. Thus there exists a continuum of mutually non-isomorphic ROG cones defined by subspaces $L \subset \mathcal{S}^6$ of type (3).

The cone $K_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$ is obtained from the face $\mathcal{F}_6(H_0) \cong \mathcal{S}_+^2$ by consecutive intertwining with the faces $\mathcal{F}_6(H_i) \cong \mathcal{S}_+^2$, $i = 1, \dots, 4$. It is hence an intertwining of 5 full matrix cones \mathcal{S}_+^2 . More complicated ROG cones can be obtained by starting with a matrix cone \mathcal{S}_+^n and consecutively intertwining it with matrix cones $\mathcal{S}_+^{k_1}, \dots, \mathcal{S}_+^{k_m}$ along full faces of ranks d_1, \dots, d_m , where $1 \leq d_i < \min(n, k_i)$, $i = 1, \dots, m$. In this way, families of mutually non-isomorphic ROG cones can be obtained which are parameterized by an arbitrary number of real parameters. Note that all cones obtained in such a way are simple.

6 Dimension and degree of ROG cones

In this section we consider the relation between the dimension and the degree of a ROG cone K . Evidently we have the inequality chain $\deg K \leq \dim K \leq \frac{\deg K(\deg K + 1)}{2}$, with equality on the left if and only if K is isomorphic to the cone of positive semi-definite diagonal matrices, and equality on the right if and only if K is isomorphic to the full cone of positive semi-definite matrices. We shall say that a ROG cone K has *codimension* k if $\dim K = \frac{\deg K(\deg K + 1)}{2} - k$. The codimension can be interpreted as the number of linearly independent linear constraints on the matrices $X \in K$ in any non-degenerate representation of K .

Lemma 6.1. *Let K be a ROG cone of degree n and dimension $\frac{n(n+1)}{2} - k$. Then K has a representation $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_i \rangle = 0 \ \forall i = 1, \dots, k\}$, where Q_1, \dots, Q_k are linearly independent quadratic forms on \mathbb{R}^n such that every nonzero form in the linear span of $\{Q_1, \dots, Q_k\}$ is indefinite.*

Proof. By Lemma 2.5 and Corollary 3.8 there exists a non-degenerate representation of K as a linear section of \mathcal{S}_+^n . We have $\dim \mathcal{S}^n - \dim K = k$, and the orthogonal complement of $\text{span } K$ in the space of quadratic forms on \mathbb{R}^n has dimension k . Let $\{Q_1, \dots, Q_k\}$ be a basis of this complement. Then by construction we have $K = \text{span } K \cap \mathcal{S}_+^n = \{X \in \mathcal{S}_+^n \mid \langle X, Q_i \rangle = 0 \ \forall i = 1, \dots, k\}$.

Let $X \in K$ be a positive definite matrix. Suppose for the sake of contradiction that there exists a nonzero linear combination Q of Q_1, \dots, Q_k which is semi-definite. By possibly replacing Q by $-Q$, we may assume that Q is positive semi-definite. Then $\langle Q, X \rangle > 0$, leading to a contradiction. \square

In the next subsections we classify ROG cones of codimensions 1 and 2, and give a lower bound on the dimension of simple cones K of fixed degree.

6.1 ROG cones of codimension 1

In this subsection we show that all spectrahedral cones of codimension 1 are ROG. This result is closely linked to Dines' and Brickmans theorems on the convexity of the numerical range of quadratic forms [5],[3]. All these results are based on the following dimensional argument.

Lemma 6.2. *Let $L \subset \mathcal{S}^n$ be a linear subspace of dimension $\frac{n(n+1)}{2} - d$. Then the spectrahedral cone $K = L \cap \mathcal{S}_+^n$ has no extreme elements of rank $k > -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(d+1)}$.*

Proof. Let X lie on an extreme ray of K , and let $k = \text{rk } X$. Then the minimal face of \mathcal{S}_+^n which contains X has dimension $\frac{k(k+1)}{2}$. Denote this face by F . The minimal face of K which contains X is given by the intersection $F \cap L$ and has dimension 1. But since L has codimension d , we have $1 = \dim(F \cap L) \geq \dim F - d = \frac{k(k+1)}{2} - d$. This yields $k(k+1) \leq 2(d+1)$, which implies $k \leq -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(d+1)}$. \square

Corollary 6.3. *Let $L \subset \mathcal{S}^n$ be a linear subspace of dimension $\frac{n(n+1)}{2} - 1$. Then the cone $K = L \cap \mathcal{S}_+^n$ is ROG.*

Proof. By Lemma 6.2 the cone K has no extreme elements of rank $k \geq 2 > \frac{-1+\sqrt{17}}{2}$. Thus K is ROG. \square

Corollary 6.4. *Every ROG cone of degree n and codimension 1 has a representation of the form $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q \rangle = 0\}$ for some indefinite quadratic form Q , and every cone of this form is ROG of degree n and codimension 1. Two such cones K, K' , defined by indefinite quadratic forms Q, Q' , respectively, are isomorphic if and only if either Q, Q' or $Q, -Q'$ have the same signature.*

Proof. The first claim follows from Lemma 6.1.

Let now Q be an indefinite quadratic form. Then the cone $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q \rangle = 0\}$ is ROG by Corollary 6.3. Since Q is indefinite, there exists a positive definite matrix X such that $\langle X, Q \rangle = 0$. Hence K intersects the interior of \mathcal{S}_+^n , and therefore $\dim K = \dim \mathcal{S}^n - 1$. Moreover, by Corollary 3.8 K is of degree n .

Let now the cones K, K' be defined by indefinite quadratic forms Q, Q' , respectively. The cones K, K' are isomorphic if and only if their linear hulls $L = \{X \in \mathcal{S}^n \mid \langle X, Q \rangle = 0\}$, $L' = \{X \in \mathcal{S}^n \mid \langle X, Q' \rangle = 0\}$ can be taken to each other by a coordinate transformation of \mathbb{R}^n . This is the case if and only if the orthogonal complements of L, L' , namely the 1-dimensional subspaces generated by Q and Q' , are related by a coordinate transformation. The last claim now easily follows. \square

It is not hard to establish that there are $\lfloor \frac{n^2}{4} \rfloor$ isomorphism classes of ROG cones of degree n and codimension 1. For $n \geq 3$ all of them are simple.

6.2 ROG cones of codimension 2

In this subsection we classify the ROG cones K of degree n and dimension $\frac{n(n+1)}{2} - 2$. If $n = 2$, then the dimension of K is either 2 or 3, and K cannot be of codimension 2. We shall henceforth assume $n \geq 3$. For the classification we shall need the auxiliary Lemmas B.5 and B.1 which are provided in the Appendix.

Theorem 6.5. *Let K be a ROG cone of degree $n \geq 3$ and of codimension $d = 2$. Then K is isomorphic to the direct sum $\mathcal{S}_+^1 \oplus \mathcal{S}_+^2$ if $n = 3$ and to a full extension of this sum if $n > 3$.*

Proof. By Lemma 2.5 and Corollary 3.8 we may assume that K has a non-degenerate representation by matrices of size $n \times n$. By Lemma 6.1 we have $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$ for some linearly independent quadratic forms Q_1, Q_2 on \mathbb{R}^n . Since K is ROG, it has no extremal elements of rank 2 and hence the assumptions of Lemma B.5 in the Appendix are fulfilled.

Suppose that condition (i) of this lemma holds. Then for every $z \in \mathbb{R}^n$ such that zz^T is an extreme element of K , the linear forms $Q_1 z, Q_2 z$ are linearly dependent. This implies that z is an eigenvector of the pencil $Q_1 + \lambda Q_2$.

Since the degree of K is n , by Corollary 3.16 there exist n linearly independent vectors $z_1, \dots, z_n \in \mathbb{R}^n$ such that the rank 1 matrices $z_k z_k^T$ are in K for $k = 1, \dots, n$. This implies that the pencil $Q_1 + \lambda Q_2$ has n linearly independent real eigenvectors. Therefore the conditions of Lemma B.1 are satisfied. Let $\mathbb{R}^n = H_0 \oplus H_1 \oplus \dots \oplus H_m$ be the direct sum decomposition from this lemma. If $m \leq 1$, then the forms Q_1, Q_2 are linearly dependent, which contradicts our assumptions. Hence $m \geq 2$.

Let $x_1 \in H_1, x_2 \in H_2$ be nonzero vectors. Consider the matrix $X = x_1 x_2^T + x_2 x_1^T \in \mathcal{S}^n$. We have $\langle Q_i, X \rangle = 2x_1^T Q_i x_2 = 0$ for $i = 1, 2$, and hence $X \in \text{span } K$. On the other hand, $\text{span } K$ is generated by all rank 1 matrices in K because K is ROG. However, if $z \in \mathbb{R}^n$ is such that $zz^T \in K$, then by Lemma B.1 we have $z \in \bigcup_{k=1}^m (H_0 + H_k)$. It follows that $\text{span } K \subset \sum_{k=1}^m \mathcal{L}_n(H_0 + H_k)$. But $X \notin \sum_{k=1}^m \mathcal{L}_n(H_0 + H_k)$, leading to a contradiction.

Thus condition (ii) of Lemma B.5 holds. By choosing an appropriate basis of \mathbb{R}^n , we can assume that the linear forms u, q_1, q_2 from this lemma are the first elements of the dual basis. Then the cone K is given by the set $\{X \in \mathcal{S}_+^n \mid X_{12} = X_{13} = 0\}$. The claim of the theorem now easily follows. \square

Hence there is only one isomorphism class of ROG cones of codimension 2 for a given degree $n \geq 3$, in contrast to ROG cones of codimension 1, of which there are many.

6.3 Lower bound on the dimension of simple ROG cones

In this section we show that for simple ROG cones K the dimension of K is bounded from below by $2 \cdot \deg K - 1$. This will be useful later for the classification of simple ROG cones of low degree. We shall need the following auxiliary result.

Lemma 6.6. *Let $x_1, \dots, x_m \in \mathbb{R}^n$ be linearly independent vectors, and let $S \subset \mathcal{S}^n$ be the m -dimensional subspace spanned by the rank 1 matrices $x_1 x_1^T, \dots, x_m x_m^T$. Let further $H \subset \mathbb{R}^n$ be a linear subspace. Then the dimension of the intersection $S \cap \mathcal{L}_n(H)$ is given by the number of indices i such that $x_i \in H$. In particular, $\dim(S \cap \mathcal{L}_n(H)) \leq \dim H$.*

Proof. Define the index set $I = \{i \mid x_i \in H\}$. Let $A = \sum_{i=1}^m \alpha_i x_i x_i^T$ be an arbitrary element of S , where α_i are scalar coefficients. Suppose there exists an index $j \notin I$ such that $\alpha_j \neq 0$. Let $y \in \mathbb{R}^n$ be a vector such that $y^T x_j = 1$, and $y^T x_i = 0$ for all $i \neq j$. Such a vector y exists by the linear independence of x_1, \dots, x_m . We then get $Ay = \sum_{i=1}^m \alpha_i (y^T x_i) x_i = \alpha_j x_j \notin H$. Hence $A \notin \mathcal{L}_n(H)$.

It follows that every matrix in the intersection $S \cap \mathcal{L}_n(H)$ is of the form $A = \sum_{i \in I} \alpha_i x_i x_i^T$ for some scalars α_i . On the other hand, for every such matrix A and every vector $y \in \mathbb{R}^n$ we have $Ay = \sum_{i \in I} \alpha_i (y^T x_i) x_i \in H$, and $A \in \mathcal{L}_n(H)$. Therefore the intersection $S \cap \mathcal{L}_n(H)$ equals the linear span of the set $\{x_i x_i^T \mid i \in I\}$. The claims of the lemma now easily follow. \square

Theorem 6.7. *Let K be a simple ROG cone of degree n . Then $\dim K \geq 2n - 1$.*

Proof. Represent K as a linear section of \mathcal{S}_+^n . Recall that by Lemma 3.10 every face of K is a ROG cone, and that K itself is the face of K of largest degree n . Denote by \mathbf{F} the set of faces F of K such that $\dim F \geq 2 \deg F - 1$. The set \mathbf{F} is not empty, because every extreme ray of K is an element of \mathbf{F} . Set $k = \max_{F \in \mathbf{F}} \deg F$. Assume for the sake of contradiction that $K \notin \mathbf{F}$, and hence $k < n$. Let $F_k \in \mathbf{F}$ be a face of K which achieves the maximal degree k . Denote the linear span of K by L , and the linear span of F_k by L_k . By construction we have $\dim L_k \geq 2k - 1$.

By Corollary 3.8 the maximal rank of matrices in F_k equals k . Let $Y \in F_k$ be a matrix of maximal rank k , and let the k -dimensional subspace $H \subset \mathbb{R}^n$ be its image. Then we have $L_k = L \cap \mathcal{L}_n(H)$ and $F_k = L \cap \mathcal{F}_n(H)$. By Corollary 3.15 there exists a basis $\{r_1, \dots, r_k\}$ of H such that $r_i r_i^T \in K$ for all $i = 1, \dots, k$, and $Y = \sum_{i=1}^k r_i r_i^T$. By virtue of $\deg K = n$ and Corollary 3.16 we may complete this

basis of H to a basis $\{r_1, \dots, r_n\}$ of \mathbb{R}^n such that $r_i r_i^T \in K$ for all $i = 1, \dots, n$. Adopt the coordinate system defined by this basis. Then all diagonal matrices are in L , and the subspace L_k consists of the matrices in L all whose non-zero elements are located in the upper left $k \times k$ block.

Since K is simple, there exists a rank 1 matrix $zz^T \in K$ such that the vector $z = (z_1, \dots, z_n)^T$ is neither in H nor in $\text{span}\{r_{k+1}, \dots, r_n\}$. In other words, the subvector $z_H = (z_1, \dots, z_k)^T$ is not zero, and not all of the elements z_{k+1}, \dots, z_n are zero. Without loss of generality, let the nonzero elements in the second group be z_{k+1}, \dots, z_{k+m} . By scaling the vector z , we may also assume that $z^T z = 1$.

Denote by F_{k+m} the face of K which consists of all matrices in K whose non-zero elements are located in the upper left $(k+m) \times (k+m)$ block. Denote the linear span of F_{k+m} by L_{k+m} . Since all diagonal matrices are in L , the maximal rank of the matrices in F_{k+m} equals $k+m$. By Corollary 3.8 we get $\deg F_{k+m} = k+m > k$. By our definition of k we then have $F_{k+m} \notin \mathbf{F}$, and hence $\dim L_{k+m} < 2(k+m) - 1$. Let S be the $(\dim L_k + m)$ -dimensional subspace of L_{k+m} spanned by L_k and the rank 1 matrices $r_{k+1} r_{k+1}^T, \dots, r_{k+m} r_{k+m}^T$.

We have $zz^T \in F_{k+m}$. Consider the matrix $X = \text{diag}(I_{k+m}, 0, \dots, 0) - zz^T \in L_{k+m}$, where I_{k+m} is the $(k+m) \times (k+m)$ identity matrix. By $z^T z = 1$ the matrix X is positive semi-definite of rank $k+m-1$, with z as kernel vector. It follows that $X \in F_{k+m}$, and by Corollary 3.15 there exist $k+m-1$ linearly independent vectors $x_1, \dots, x_{k+m-1} \in \mathbb{R}^n$ such that $x_i x_i^T \in F_{k+m}$ for all $i = 1, \dots, k+m-1$, and $X = \sum_{i=1}^{k+m-1} x_i x_i^T$. Since $z^T X z = \sum_{i=1}^{k+m-1} (z^T x_i)^2 = 0$, it follows that $z^T x_i = 0$ for all $i = 1, \dots, k+m-1$.

Consider the $(k+m-1)$ -dimensional subspace $S' \subset L_{k+m}$ spanned by the rank 1 matrices $x_i x_i^T$, $i = 1, \dots, k+m-1$. Let us bound the dimension of the intersection $S \cap S'$. Let $A \in S \cap S'$ be arbitrary. Since $A \in S$, the matrix A has a block-diagonal structure $A = \text{diag}(A_H, a_{k+1}, \dots, a_{k+m}, 0, \dots, 0)$, with A_H a block of size $k \times k$. On the other hand, $A \in S'$ implies $Az = 0$. It follows that $a_{k+1} z_{k+1} = \dots = a_{k+m} z_{k+m} = 0$ and $a_{k+1} = \dots = a_{k+m} = 0$, because the corresponding elements of z are non-zero. The image of A is hence contained in the intersection of the subspace H with the orthogonal complement of z . By virtue of $z_H \neq 0$ this intersection has dimension $k-1$. By Lemma 6.6 we then get that $\dim(S \cap S') \leq k-1$.

Thus $\dim(S + S') = \dim S + \dim S' - \dim(S \cap S') \geq (\dim L_k + m) + (k+m-1) - (k-1) \geq 2k-1+2m$, leading to a contradiction with the bound $\dim L_{k+m} < 2(k+m) - 1$. This completes the proof. \square

7 Isolated extreme rays

The extreme rays of a ROG cone are generated by its rank 1 matrices. In this section we study the situation when an extreme ray of a ROG cone K is isolated. We shall show that in this case K is a direct sum of \mathcal{S}_+^1 and a lower-dimensional ROG cone, and the isolated extreme ray is the face of K corresponding to the factor \mathcal{S}_+^1 . We deduce a couple of results for simple ROG cones and consider the situation when a simple ROG cone K has a face of codimension 2. We will need the following concept.

Definition 7.1. The vectors $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ are called *minimally linearly dependent* if they are linearly dependent, but every k of them are linearly independent.

Lemma 7.2. A set of vectors $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ is minimally linearly dependent if and only if their span has dimension k and there exist nonzero real numbers c_1, \dots, c_{k+1} such that $\sum_{i=1}^{k+1} c_i x_i = 0$.

Proof. Denote by L the linear span of $\{x_1, \dots, x_{k+1}\}$, and let X be the $n \times (k+1)$ matrix formed of the column vectors x_i .

Let $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ be minimally linearly dependent. Then the dimension of L equals k , because there exist k linearly independent vectors in L . The matrix X then has rank k and its kernel has dimension 1. Let $(c_1, \dots, c_{k+1})^T \in \mathbb{R}^{k+1}$ be a generator of $\ker X$. Then $\sum_{i=1}^{k+1} c_i x_i = 0$ and not all c_i are zero. Let $I \subset \{1, \dots, k+1\}$ be the set of indices i such that $c_i \neq 0$. Then the vectors in the set $\{x_i \mid i \in I\}$ are linearly dependent. By assumption, no k vectors are linearly dependent, and therefore I has not less than $k+1$ elements. It follows that $c_i \neq 0$ for all i .

Let now c_1, \dots, c_{k+1} be nonzero real numbers such that $\sum_{i=1}^{k+1} c_i x_i = 0$, and suppose $\dim L = k$. Then x_1, \dots, x_{k+1} are linearly dependent. Moreover, $\text{rk } X = k$, and hence the vector $(c_1, \dots, c_{k+1})^T$ generates the kernel of X . In particular, there is no nonzero kernel vector with a zero element. It follows that every subset of k vectors is linearly independent. Thus x_1, \dots, x_{k+1} are minimally linearly dependent. \square

Lemma 7.3. *Let $S \subset \mathbb{R}^n$ be a subset and $x \in S$ a nonzero vector. Then either*

- 1) *there exists a subspace $H \subset \mathbb{R}^n$ of dimension $n-1$ which does not contain x , such that for every $y \in S$ either $y \in H$ or y is a multiple of x ,*
- or 2) *there exists a minimally linearly dependent subset $T \subset S$ of size at least 3 such that $x \in T$.*

Proof. Let $L \subset \mathbb{R}^n$ be the linear span of S , and let k be its dimension. Let us complete $x_1 = x$ to a basis $\{x_1, \dots, x_k\} \subset S$ of L . Then every vector $y \in S$ can be in a unique way represented as a sum $y = \sum_{i=1}^k c_i x_i$. We have two possibilities.

1) For every vector $y = \sum_{i=1}^k c_i x_i \in S$, either $c_1 = 0$, or $c_2 = \dots = c_k = 0$. Then we can take H as any hyperplane which contains the span of $\{x_2, \dots, x_k\}$ but not x_1 , and are in the situation 1) of the lemma.

2) There exists $y = \sum_{i=1}^k c_i x_i \in S$ such that $c_1 \neq 0$ and at least one of the coefficients c_2, \dots, c_k is not zero. Let without loss of generality the nonzero coefficients among the c_2, \dots, c_k be the coefficients c_2, \dots, c_l , $l \geq 2$. Then we obtain $y - \sum_{i=1}^l c_i x_i = 0$, and the set $\{x_1, \dots, x_l, y\} \subset S$ is minimally linearly dependent by Lemma 7.2. Thus we are in the situation 2) of the lemma. \square

Lemma 7.4. *Let K be a ROG cone and let $R_1, \dots, R_{k+1} \in K$ be extreme rays of K . Let the rank 1 matrices $X_i = x_i x_i^T$ be generators of these extreme rays, respectively, in some representation of K as a linear section of a positive semi-definite matrix cone S_+^n . Whether the set $\{x_1, \dots, x_{k+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent then depends only on the extreme rays R_1, \dots, R_{k+1} of K , but not on the representation of K , its size, or the generators X_i .*

Proof. Let c_1, \dots, c_{k+1} be non-zero real numbers. Then a subset $\{x_1, \dots, x_{k+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent if and only if the subset $\{c_1 x_1, \dots, c_{k+1} x_{k+1}\}$ is minimally linearly dependent. This follows directly from Definition 7.1. Hence the property does not depend on the generators X_i of the extreme rays for a given representation of K . Let now $X_i = x_i x_i^T$, $Y_i = y_i y_i^T$ be generators of the rays R_i in different representations of sizes n, m , respectively. Let $n \leq m$ without loss of generality. By Theorem 3.18 there exists an injective linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $f(x_i) = \sigma_i y_i$, where $\sigma_i \in \{-1, +1\}$, for all $i = 1, \dots, k+1$. By the injectivity of f , we have for every index subset $I \subset \{1, \dots, k+1\}$ that the set $\{x_i\}_{i \in I}$ is linearly dependent if and only if the set $\{\sigma_i y_i\}_{i \in I}$ is linearly dependent. Hence $\{x_1, \dots, x_{k+1}\}$ is minimally linearly dependent if and only if the set $\{y_1, \dots, y_{k+1}\}$ is. This completes the proof. \square

Lemma 7.4 allows to make the following definition.

Definition 7.5. Let K be a ROG cone. We call a subset $\{R_1, \dots, R_{k+1}\}$ of extreme rays of K , generated by rank 1 matrices $X_i = x_i x_i^T$, respectively, an *MLD set*, if the set $\{x_1, \dots, x_{k+1}\}$ is minimally linearly dependent.

Lemma 7.6. *Let K be a ROG cone of degree $n \geq 2$, possessing an MLD set $\{R_1, \dots, R_{n+1}\}$ of extreme rays. Then the following holds:*

- i) *the cone K is simple;*
- ii) *the extreme rays R_1, \dots, R_{n+1} of K are not isolated.*

Proof. Represent K as a linear section of S_+^n , and let the rank 1 matrices $X_i = x_i x_i^T$ be generators of the extreme rays R_i , $i = 1, \dots, n+1$. Then the set $\{x_1, \dots, x_{n+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent. Denote the linear span of K by L .

Suppose for the sake of contradiction that K is not simple. Then there exists a nontrivial direct sum decomposition $\mathbb{R}^n = H_1 \oplus H_2$ such that $K \subset \mathcal{L}_n(H_1) + \mathcal{L}_n(H_2)$ and hence $x_i \in H_1 \cup H_2$ for all $i = 1, \dots, n+1$. Let n_1, n_2 be the dimensions of H_1, H_2 , respectively, and n'_1, n'_2 the number of indices i such that $x_i \in H_1$ or $x_i \in H_2$, respectively. Then $n'_1, n'_2 > 0$, because the vectors x_1, \dots, x_{n+1} span the whole space \mathbb{R}^n and the decomposition $\mathbb{R}^n = H_1 \oplus H_2$ is nontrivial. On the other hand, we have $n_1 + n_2 = n$ and $n'_1 + n'_2 = n+1$. Hence either $n'_1 > n_1$, or $n'_2 > n_2$, and there exists a strict subset of the set $\{x_1, \dots, x_{n+1}\}$ which is linearly dependent, leading to a contradiction. This proves i).

We shall now prove ii). For $n = 2$ we have $K = S_+^2$, and the assertion is evident. Suppose $n \geq 3$.

By the definition of minimal linear dependence the vectors x_1, \dots, x_n form a basis of \mathbb{R}^n . Choose a coordinate system in which this is the canonical basis. By Lemma 7.2 there exist nonzero scalars c_1, \dots, c_{n+1} such that $\sum_{i=1}^{n+1} c_i x_i = 0$. We may normalize these scalars by a common factor to achieve $c_{n+1} = -1$. Then we have $x_{n+1} = (c_1, \dots, c_n)^T$.

The subspace $L \subset \mathcal{S}^n$ contains the $(n+1)$ -dimensional linear span \tilde{L} of the rank 1 matrices $x_i x_i^T$, $i = 1, \dots, n+1$. Let $d_1, \dots, d_n > 0$ be positive scalars, and set $d_{n+1} = -(\sum_{i=1}^n d_i^{-1} c_i^2)^{-1}$. Then the matrix $M = \sum_{i=1}^{n+1} d_i x_i x_i^T$ is an element of \tilde{L} . Moreover, for every vector $r = (r_1, \dots, r_n)^T$ we have

$$r^T M r = \sum_{i=1}^{n+1} d_i (r^T x_i)^2 = \sum_{i=1}^n d_i r_i^2 - \frac{(\sum_{i=1}^n c_i r_i)^2}{\sum_{i=1}^n d_i^{-1} c_i^2} = \sum_{i=1}^n \left(\sqrt{d_i} r_i - \frac{c_i \sum_{j=1}^n c_j r_j}{\sqrt{d_i} \sum_{j=1}^n d_j^{-1} c_j^2} \right)^2 \geq 0.$$

It follows that $M \succeq 0$ and hence $M \in K$.

Moreover, we have $r^T M r = 0$ if and only if $\sqrt{d_i} r_i = \frac{c_i \sum_{j=1}^n c_j r_j}{\sqrt{d_i} \sum_{j=1}^n d_j^{-1} c_j^2}$ for all $i = 1, \dots, n$. An equivalent condition is that $r = \alpha s$ for some scalar α , where $s = (s_1, \dots, s_n)^T$ is a vector given by $s_i = d_i^{-1} c_i$ for all $i = 1, \dots, n$. Hence M is of rank $n-1$, in particular, it is not rank 1.

Let H be the $(n-1)$ -dimensional subspace of vectors $v \in \mathbb{R}^n$ such that $v^T s = 0$. Then the minimal face of \mathcal{S}_+^n which contains M is given by $\mathcal{F}_n(H)$. It consists of all matrices $X \in \mathcal{S}_+^n$ such that $Xs = 0$. The linear span $\mathcal{L}_n(H)$ of this face is given by all $X \in \mathcal{S}^n$ such that $Xs = 0$. We shall now compute the intersection $\mathcal{L}_n(H) \cap \tilde{L}$. Let $X = \sum_{i=1}^{n+1} \alpha_i x_i x_i^T \in \mathcal{L}_n(H) \cap \tilde{L}$. Then we have

$$Xs = \sum_{i=1}^{n+1} \alpha_i (x_i^T s) x_i = \sum_{i=1}^n \alpha_i s_i x_i + \alpha_{n+1} \cdot \sum_{j=1}^n c_j s_j \cdot \sum_{i=1}^n c_i x_i = \sum_{i=1}^n \left(\alpha_i d_i^{-1} + \alpha_{n+1} \sum_{j=1}^n d_j^{-1} c_j^2 \right) c_i x_i = 0.$$

It follows that $\alpha_i d_{n+1} = \alpha_{n+1} d_i$ for all $i = 1, \dots, n$. An equivalent condition is that the vectors $\alpha = (\alpha_1, \dots, \alpha_{n+1})^T$ and $d = (d_1, \dots, d_{n+1})^T$ are proportional, and hence X is proportional to M . It follows that $\mathcal{L}_n(H) \cap \tilde{L}$ is the 1-dimensional subspace generated by M .

By Lemma 3.13 there exist $n-1$ linearly independent vectors $y_1, \dots, y_{n-1} \in \mathbb{R}^n$ such that $y_i y_i^T \in K$ for all i and $M = \sum_{i=1}^{n-1} y_i y_i^T$. Note that $y_i y_i^T \in \mathcal{L}_n(H)$ for all i , and hence $y_i y_i^T \in \mathcal{L}_n(H) \cap L$.

Assume for the sake of contradiction that the extreme ray R_1 generated by the rank 1 matrix $x_1 x_1^T$ is an isolated extreme ray of K . Then there exists $\beta > 0$ such that for every vector $z \in \mathbb{R}^n$, not proportional to x_1 and such that $zz^T \in K$, we have $\sum_{i=2}^n (z^T x_i)^2 > \beta (z^T x_1)^2$.

Since the intersection $\mathcal{L}_n(H) \cap \tilde{L}$ does not contain a rank 1 matrix, $x_1 x_1^T \in \tilde{L}$, and $y_i y_i^T \in \mathcal{L}_n(H)$, we have that y_i is not proportional to x_1 for every $i = 1, \dots, n-1$. It follows that $\sum_{j=2}^n (y_i^T x_j)^2 > \beta (y_i^T x_1)^2$ for all $i = 1, \dots, n-1$. Therefore

$$\sum_{j=2}^n (d_j + d_{n+1} c_j^2) = \sum_{j=2}^n x_j^T M x_j = \sum_{i=1}^{n-1} \sum_{j=2}^n (x_j^T y_i)^2 > \beta \sum_{i=1}^{n-1} (y_i^T x_1)^2 = \beta x_1^T M x_1 = \beta (d_1 + d_{n+1} c_1^2). \quad (5)$$

Fix now d_2, \dots, d_n and let $d_1 \rightarrow +\infty$. Then $d_{n+1} \rightarrow -(\sum_{i=2}^n d_i^{-1} c_i^2)^{-1}$, and the leftmost term in (5) tends to a finite value. On the other hand, the rightmost term in (5) tends to $+\infty$, leading to a contradiction.

For the other extreme rays of K the reasoning is similar after an appropriate permutation of the MLD set $\{R_1, \dots, R_{n+1}\}$. \square

Corollary 7.7. *Let $k \geq 2$ and let K be a ROG cone possessing an MLD set $\{R_1, \dots, R_{k+1}\}$ of extreme rays. Then the following holds:*

- i) *the dimension and degree of K satisfy $\dim K \geq 2k-1$, $\deg K \geq k$;*
- ii) *the extreme rays R_1, \dots, R_{k+1} of K are not isolated.*

Proof. Represent K as a linear section of \mathcal{S}_+^n for some n , and let the rank 1 matrices $X_i = x_i x_i^T$ be generators of the extreme rays R_i , $i = 1, \dots, k+1$. Then the set $\{x_1, \dots, x_{k+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent. By Lemma 7.2 the linear span H of the vectors x_1, \dots, x_{k+1} is a subspace of dimension k . Then $K_H = \mathcal{L}_n(H) \cap K$ is a face of K and hence a ROG cone by Lemma 3.10. Moreover, K_H contains the rank 1 matrices $x_1 x_1^T, \dots, x_{k+1} x_{k+1}^T$ and is of degree k . In particular, the set $\{R_1, \dots, R_{k+1}\}$ is also an MLD set of extreme rays for K_H .

Applying Lemma 7.6 to the cone K_H , we see that K_H is simple and the extreme rays R_1, \dots, R_{k+1} of K_H are not isolated for all $i = 1, \dots, k+1$. By Theorem 6.7 we have $\dim K_H \geq 2k-1$. But $\dim K \geq \dim K_H$, $\deg K \geq \deg K_H$, and every extreme ray of K_H is also an extreme ray of K . The claim of the corollary now easily follows. \square

Corollary 7.8. *Let K be a ROG cone of degree n , and let R be an isolated extreme ray of K . Then K can be represented as a direct sum $K' \oplus \mathcal{S}_+^1$, where K' is a ROG cone of degree $n - 1$, such that the extreme ray R is given by the set $\{0\} \oplus \mathcal{S}_+^1$.*

Proof. Represent K as a linear section of the cone \mathcal{S}_+^n , and let $x \in \mathbb{R}^n$ be such that $X = xx^T$ generates the isolated extreme ray R of K .

Define the set $S = \{y \in \mathbb{R}^n \mid yy^T \in K\}$ and note that $x \in S$. By virtue of Corollary 7.7 the vector x cannot be contained in a minimally linearly dependent subset of S of cardinality at least 3. By Lemma 7.3 there exists a subspace $H \subset \mathbb{R}^n$ of dimension $n - 1$ such that $x \notin H$ and $S \subset H \cup \text{span}\{x\}$.

Hence $\text{span } K = \text{span}\{yy^T \mid y \in S\} \subset \mathcal{L}_n(H) + \text{span } R$, and by Lemma 4.7 we have $K = K' + R$, where $K' = K \cap \mathcal{L}_n(H)$ is the face of K generated by H , and the sum is isomorphic to the direct sum of the summands. By Lemma 4.4 the cone K' has degree $n - 1$. This completes the proof. \square

Theorem 7.9. *Let K be a ROG cone of degree n . Then the number of its isolated extreme rays does not exceed n . Let R_1, \dots, R_k be the isolated extreme rays of K . Then K is isomorphic to a direct sum $K' \oplus \mathbb{R}_+^k$, where K' is a ROG cone of degree $n - k$ without isolated extreme rays, and the extreme rays R_1, \dots, R_k correspond to the extreme rays of the summand \mathbb{R}_+^k .*

Proof. We prove the theorem by induction over n . If $n = 1$, then $K = \mathbb{R}_+$, and the assertion is evident. Suppose now that $n \geq 2$ and the assertion is proven for cones of degrees not exceeding $n - 1$.

If K has no isolated extreme ray, then the assertion of the theorem holds with $K' = K$.

Assume now that R is an isolated extreme ray of K . By Corollary 7.8 K can be represented as a direct sum $K_1 \oplus \mathbb{R}_+$, where K_1 is a ROG cone of degree $n - 1$. By the assumption of the induction, the number of isolated extreme rays of K_1 is finite and does not exceed $n - 1$, let these be $\rho_2, \dots, \rho_{k'}$, $1 \leq k' \leq n$. Moreover, K_1 is isomorphic to a direct sum $K' \oplus \mathbb{R}_+^{k'-1}$, where K' is a ROG cone of degree $n - k'$ without isolated extreme rays. It follows that $K \cong K' \oplus \mathbb{R}_+^{k'}$.

Now every extreme ray of the direct sum $K' \oplus \mathbb{R}_+^{k'}$ is either an extreme ray of the factor K' or an extreme ray of the factor $\mathbb{R}_+^{k'}$, and it is isolated in the direct sum if and only if it is isolated in the factor. The extreme rays of K' are not isolated in K' , and hence they are not isolated in K . The factor $\mathbb{R}_+^{k'}$ has k' extreme rays, and all of them are isolated. These k' rays hence exhaust the isolated extreme rays of $K' \oplus \mathbb{R}_+^{k'}$. It follows that $k' = k$ and the assertion of the theorem readily follows. \square

The discrete and the continuous part of the set of extreme rays of K thus generate separate factors of the cone K . The factor generated by the discrete part is isomorphic to the nonnegative orthant, with the discrete extreme rays of K being its generators.

Corollary 7.10. *Let K be a simple ROG cone of degree $\deg K \geq 2$. Then K has no isolated extreme rays.*

Proof. The corollary is an immediate consequence of Theorem 7.9. \square

Lemma 7.11. *Let $K \subset \mathcal{S}_+^n$ be a ROG cone, and let $x \in \mathbb{R}^n$ be such that the rank 1 matrix xx^T generates an extreme ray of K which is not isolated. Then there exists a vector $y \in \mathbb{R}^n$, linearly independent of x , such that $xy^T + yx^T \in \text{span } K$.*

Proof. Assume the conditions of the lemma. Then there exists a sequence v_1, v_2, \dots of nonzero vectors in \mathbb{R}^n such that $x^T v_k = 0$, $(x + v_k)(x + v_k)^T \in K$ for all k , and $\lim_{k \rightarrow \infty} v_k = 0$. Set $y_k = \frac{v_k}{\|v_k\|}$. Then we have $\frac{(x+v_k)(x+v_k)^T - xx^T}{\|v_k\|} = xy_k^T + y_k x^T + \frac{v_k v_k^T}{\|v_k\|} \in \text{span } K$. Since $\lim_{k \rightarrow \infty} \frac{v_k v_k^T}{\|v_k\|} = 0$ and $\text{span } K$ is closed, we have $xy^T + yx^T \in \text{span } K$ for every accumulation point of the sequence y_1, y_2, \dots . But such accumulation points exist due to the compactness of the unit sphere, and every such accumulation point is orthogonal to x . This completes the proof. \square

Corollary 7.12. *Let $K \subset \mathcal{S}_+^n$ be a simple ROG cone of degree $\deg K \geq 2$. Then for every nonzero vector $x \in \mathbb{R}^n$ such that $xx^T \in K$ there exists a vector $y \in \mathbb{R}^n$, linearly independent of x , such that $xy^T + yx^T \in \text{span } K$.*

Proof. The corollary is an immediate consequence of Lemma 7.11 and Corollary 7.10. \square

Lemma 7.13. *Let K be a simple ROG cone of degree $\deg K \geq 2$. If K has a face $F \subset K$ such that $\dim K - \dim F = 2$, then K is isomorphic to an intertwining of F and \mathcal{S}_+^2 .*

Proof. Assume the conditions of the lemma, and set $n = \deg K$, $k = \deg F$. Represent K as a linear section of the cone \mathcal{S}_+^n , and let $X \in F$ be a matrix of maximal rank k . Denote the image of X by H . Then $F = \mathcal{L}_n(H) \cap K$. By Corollary 3.16 there exist linearly independent vectors $r_{k+1}, \dots, r_n \in \mathbb{R}^n$ such that $\mathbb{R}^n = \text{span}(H \cup \{r_{k+1}, \dots, r_n\})$ and $r_j r_j^T \in K$, $j = k+1, \dots, n$. We obtain $\dim K \geq \dim F + \dim \text{span}\{r_{k+1} r_{k+1}^T, \dots, r_n r_n^T\} = (\dim K - 2) + (n - k)$. It follows that $k \geq n - 2$. If $k = n - 2$, then $\text{span } K = \text{span } F + \text{span}\{r_{n-1} r_{n-1}^T, r_n r_n^T\}$ and K is isomorphic to the direct sum $F \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, contradicting the simplicity of K .

Hence $k = n - 1$. Set $x = r_n$ for simplicity of notation. By Corollary 7.12 there exists a nonzero vector $y \in H$ such that $xy^T + yx^T \in \text{span } K$.

Since the codimension of F in K is two, we have $\text{span } K = \text{span } F \oplus \text{span } xx^T \oplus \text{span}(xy^T + yx^T)$. Since K is simple, there exists a vector $z \in \mathbb{R}^n$ such that $z \notin H \cup \text{span}\{x\}$ and $zz^T \in K$. Let $z = z_H + \beta x$ be the decomposition of z corresponding to the direct sum decomposition $\mathbb{R}^n = H \oplus \text{span}\{x\}$. Then $z_H \neq 0$, $\beta \neq 0$, $zz^T = z_H z_H^T + \beta(z_H x^T + x z_H^T) + \beta^2 xx^T$. On the other hand, we have the decomposition $zz^T = Z_F + \alpha_1 xx^T + \alpha_2(xy^T + yx^T)$, where $Z_F \in \text{span } F$.

Let l be a linear form which is zero on H , but $l(x) = 1$. Contracting both decompositions of the rank 1 matrix zz^T with l , we obtain $\beta z_H + \beta^2 x = \alpha_1 x + \alpha_2 y$, and hence $\alpha_1 = \beta^2$, $\beta z_H = \alpha_2 y$, $\alpha_2 \neq 0$, $Z_F = z_H z_H^T = (\beta^{-1} \alpha_2)^2 yy^T \in \text{span } F$.

Hence $yy^T \in F \subset K$. Thus $\mathcal{L}_n(\text{span}\{x, y\}) \subset \text{span } K$ and $\mathcal{F}_n(\text{span}\{x, y\}) = \mathcal{L}_n(\text{span}\{x, y\}) \cap K$ is a face of K which is isomorphic to \mathcal{S}_+^2 . By construction K is an intertwining of the faces F and $\mathcal{F}_n(\text{span}\{x, y\})$, with the intersection $F \cap \mathcal{F}_n(\text{span}\{x, y\})$ generated by yy^T . This yields the assertion of the lemma. \square

8 Classification for small degrees

In this section we classify all simple ROG cones K of degree $n = \deg K \leq 4$ up to isomorphism. As we already noted in Subsection 5.2, for degree 6 there exist infinitely many isomorphism classes of simple ROG cones. Whether the classification for degree 5 is finite remains an open question. Denote by Tri_+^n the cone of all tri-diagonal matrices in \mathcal{S}_+^n .

8.1 Cones of degree $n \leq 3$

For $n = 1$ the only ROG cone is \mathcal{S}_+^1 .

For $n = 2$ we have the ROG cones $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1$ and \mathcal{S}_+^2 , of which only the latter is simple.

For $n = 3$ the only ROG cone of dimension 6 is \mathcal{S}_+^3 , which is simple. By Theorem 6.7 any other simple ROG cone must have dimension 5, i.e., is given by $K = \{X \in \mathcal{S}_+^3 \mid \langle X, Q \rangle = 0\}$ for some indefinite quadratic form Q . The isomorphism class of K depends only on the signature of Q , and the forms $\pm Q$ define the same cone K . Moreover, every cone K of this form is ROG by Corollary 6.3. The possible isomorphism classes are hence given by the signatures $(++-)$ and $(+-0)$ of Q . It is easily seen that the corresponding ROG cones are isomorphic to Han_+^3 and the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, respectively. The latter cone is isomorphic to Tri_+^3 .

8.2 Cones of degree 4 and codimension $d \leq 2$

Let K be a ROG cone of degree $\deg K = 4$.

If $\dim K = 10$, then $K \simeq \mathcal{S}_+^4$.

If $\dim K = 9$, then K is of the form $\{X \in \mathcal{S}_+^4 \mid \langle X, Q \rangle = 0\}$ for some indefinite quadratic form Q . As in the case $n = 3$, the isomorphism class of K is defined by the signature of Q , where $\pm Q$ yield the same cone K . The possible isomorphism classes of K are then defined by the signatures $(+-00)$, $(++-0)$, $(++--)$, and $(++++)$ of Q . In the first two cases K is a full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1$ and Han_+^3 , respectively. In the third case K is isomorphic to the cone of positive semi-definite 4×4 block-Hankel matrices $\text{Han}_+^{2,2}$. It can be interpreted as the moment cone of the homogeneous biquadratic forms on $\mathbb{R}^2 \times \mathbb{R}^2$. It is not hard to see that all four isomorphism classes consist of simple cones.

Let $\dim K = 8$. If K is simple, then by Theorem 6.5 it is isomorphic to a full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^2$.

By Theorem 6.7 any other simple ROG cone of degree 4 must have dimension 7.

8.3 Cones of degree 4 and dimension 7

The simple ROG cones of degree 4 and dimension 7 are somewhat more difficult to classify. We shall first consider a number of special cases and then show that the general case can be reduced to one of these special cases. The most complicated case is that of cones isomorphic to the 4×4 positive semi-definite Hankel matrices, its consideration can be found in the Appendix.

Lemma 8.1. *Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of dimension 7 and degree 4. Suppose that the subspace of block-diagonal matrices consisting of two blocks of size 2×2 each is contained in $\text{span } K$. Then K is isomorphic to the cone Tri_+^4 of positive semi-definite tri-diagonal matrices.*

Proof. The subspace of block-diagonal matrices as defined in the formulation of the lemma is 6-dimensional. Hence there exist scalars $a_{13}, a_{14}, a_{23}, a_{24}$, not all equal zero, such that the linear span of K is given by all matrices of the form

$$A = \sum_{i=1}^7 \alpha_i A_i = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_7 a_{13} & \alpha_7 a_{14} \\ \alpha_2 & \alpha_3 & \alpha_7 a_{23} & \alpha_7 a_{24} \\ \alpha_7 a_{13} & \alpha_7 a_{23} & \alpha_4 & \alpha_5 \\ \alpha_7 a_{14} & \alpha_7 a_{24} & \alpha_5 & \alpha_6 \end{pmatrix}, \quad \alpha_1, \dots, \alpha_7 \in \mathbb{R},$$

where the matrices A_1, \dots, A_7 are defined by the above identity. Since K is ROG, there exists a rank 1 matrix $Z = zz^T = \sum_{i=1}^7 \zeta_i A_i$ with $\zeta_7 \neq 0$. Its upper right 2×2 block is also rank 1. Hence $a_{13}a_{24} = a_{14}a_{23}$ and there exist angles φ_1, φ_2 and a positive scalar r such that $a_{13} = r \cos \varphi_1 \cos \varphi_2$, $a_{14} = r \cos \varphi_1 \sin \varphi_2$, $a_{23} = r \sin \varphi_1 \cos \varphi_2$, $a_{24} = r \sin \varphi_1 \sin \varphi_2$. Define a basis of \mathbb{R}^4 by the vectors $x_1 = (-\sin \varphi_1, \cos \varphi_1, 0, 0)^T$, $x_2 = (\cos \varphi_1, \sin \varphi_1, 0, 0)^T$, $x_3 = (0, 0, \cos \varphi_2, \sin \varphi_2)^T$, $x_4 = (0, 0, -\sin \varphi_2, \cos \varphi_2)^T$. In the coordinates given by this basis K equals the cone Tri_+^4 , which proves our claim. \square

Lemma 8.2. *Let $K \subset \mathcal{S}_+^n$ be a ROG cone, let e_1, \dots, e_n be the canonical basis vectors of \mathbb{R}^n , and let $y = (0, y_2, \dots, y_n)^T \in \mathbb{R}^n$ be a vector such that $y_2, \dots, y_n \neq 0$. If $e_1 e_1^T, \dots, e_n e_n^T, e_1 y^T + y e_1^T \in \text{span } K$, then K is simple and $\dim K \geq 2n - 1$.*

Proof. Suppose for the sake of contradiction that K is not simple. Then there exists a nontrivial direct sum decomposition $\mathbb{R}^n = H_1 \oplus H_2$ such that for every rank 1 matrix $xx^T \in K$ we have either $x \in H_1$ or $x \in H_2$. Hence $e_i \in H_1 \cup H_2$ for $i = 1, \dots, n$. It follows that H_1, H_2 are spanned by complementary subsets of the canonical basis of \mathbb{R}^n . Hence there exists a permutation of the basis vectors such that in the corresponding coordinate system every matrix in K , and hence also in $\text{span } K$, becomes block-diagonal with a nontrivial block structure. But this is in contradiction with the assumption $e_1 y^T + y e_1^T \in \text{span } K$. Hence K must be simple.

Since the identity matrix is an element of K , we have $\deg K = n$. The bound on the dimension now follows from Theorem 6.7. \square

Corollary 8.3. *Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of dimension 7 and degree 4. Suppose there exist linearly independent vectors $z_1, z_2, z_3 \in \mathbb{R}^4$ and nonzero scalars α, β such that $z_1 z_1^T, z_2 z_2^T, z_3 z_3^T, \alpha(z_1 z_2^T + z_2 z_1^T) + \beta(z_1 z_3^T + z_3 z_1^T) \in \text{span } K$. Then K is isomorphic to either Tri_+^4 , or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or an intertwining of Han_+^3 and \mathcal{S}_+^2 .*

Proof. Denote by $H \subset \mathbb{R}^4$ the hyperplane spanned by z_1, z_2, z_3 . The face $F = \mathcal{L}_4(H) \cap K$ of K is a ROG cone by Lemma 3.10. Applying Lemma 8.2 to F , we obtain $\dim F \geq 5$.

However, $\dim F \neq 6$, because a ROG cone K with a face F of codimension 1 in K is isomorphic to $F \oplus \mathcal{S}_+^1$ and hence not simple. Therefore F has codimension 2 in K . By Lemma 7.13 the cone K is isomorphic to an intertwining of F with \mathcal{S}_+^2 .

In the previous subsection we established that a simple ROG cone of dimension 5 and degree 3 is isomorphic to either Han_+^3 or Tri_+^3 . If $F \cong \text{Han}_+^3$, then K is isomorphic to an intertwining of Han_+^3 and \mathcal{S}_+^2 . If $F \cong \text{Tri}_+^3$, then there exist two possibilities for K , because Tri_+^3 has two non-isomorphic types of extreme rays. It is not hard to see that an intertwining of Tri_+^3 with \mathcal{S}_+^2 along these two types of extreme rays leads to cones which are isomorphic to Tri_+^4 or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, respectively. \square

Lemma 8.4. *Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of dimension 7 and degree 4. Suppose that K has a face which is isomorphic to \mathcal{S}_+^2 . Then K fulfills the conditions of Corollary 8.3.*

Proof. By assumption there exist linearly independent vectors $x_1, x_2 \in \mathbb{R}^4$ such that $x_1x_1^T, x_2x_2^T, x_1x_2^T + x_2x_1^T \in \text{span } K$. By Corollary 3.16 we may complete x_1, x_2 with vectors x_3, x_4 to a basis of \mathbb{R}^4 such that $x_3x_3^T, x_4x_4^T \in K$. Pass to the coordinate system defined by this basis. By Corollary 7.12 there exists a nonzero vector $y = (y_1, y_2, 0, y_4)^T$ such that $x_3y^T + yx_3^T \in \text{span } K$.

If $y_1 = y_2 = 0$, then $x_3x_4^T + x_4x_3^T \in \text{span } K$ and K fulfills the conditions of Lemma 8.1. Hence K is isomorphic to Tri_+^4 . The claim of the lemma then immediately follows in this case.

Suppose now that y_1, y_2 are not simultaneously zero.

Let us first consider the case $y_4 = 0$. Let $F = \mathcal{L}_4(\text{span}\{x_1, x_2, x_3\}) \cap K$ be the face of K which consists of matrices $X \in K$ whose last column vanishes. Then $x_1x_1^T, x_2x_2^T, x_1x_2^T + x_2x_1^T, x_3x_3^T, x_3y^T + yx_3^T \in \text{span } F$, and $\dim F \geq 5$. Since $\dim F = 6$ is not possible by the simplicity of K , we then must have $\text{span } F = \text{span}\{x_1x_1^T, x_2x_2^T, x_1x_2^T + x_2x_1^T, x_3x_3^T, x_3y^T + yx_3^T\}$. It follows that F is isomorphic to Tri_+^3 . From Lemma 7.13 it follows that K is isomorphic to an intertwining of Tri_+^3 and \mathcal{S}_+^2 , which proves the claim of the lemma in this case.

Suppose now that $y_4 \neq 0$. Define the nonzero vector $z_3 = (y_1, y_2, 0, 0)^T$. Then $z_3z_3^T \in K$ and $y = \alpha x_4 + \beta z_3$ with $\alpha = y_4$ and $\beta = 1$. The linearly independent vectors $z_1 = x_3, z_2 = x_4, z_3$, and scalars α, β then satisfy the conditions of Corollary 8.3. \square

We are now in a position to consider the general case.

Theorem 8.5. *Let K be a simple ROG cone of degree $\deg K = 4$ and dimension $\dim K = 7$. Then K is isomorphic to either Tri_+^4 , or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or an intertwining of Han_+^3 and \mathcal{S}_+^2 , or Han_+^4 .*

Proof. Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of degree 4 and dimension 7. By Corollary 3.16 there exist linearly independent vectors x_1, x_2, x_3, x_4 such that $x_ix_i^T \in K, i = 1, \dots, 4$. Pass to the coordinate system defined by the basis $\{x_1, x_2, x_3, x_4\}$. Then all diagonal matrices are in the linear span of K . Moreover, by Corollary 7.12 there exist nonzero vectors $y_i = (y_{i1}, y_{i2}, y_{i3}, y_{i4})^T$ such that $x_i^T y_i = y_{ii} = 0$ and $x_i y_i^T + y_i x_i^T \in \text{span } K, i = 1, 2, 3, 4$. Therefore $\text{span } K$ contains all matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_5 y_{12} + \alpha_6 y_{21} & \alpha_5 y_{13} + \alpha_7 y_{31} & \alpha_5 y_{14} + \alpha_8 y_{41} \\ \alpha_5 y_{12} + \alpha_6 y_{21} & \alpha_2 & \alpha_6 y_{23} + \alpha_7 y_{32} & \alpha_6 y_{24} + \alpha_8 y_{42} \\ \alpha_5 y_{13} + \alpha_7 y_{31} & \alpha_6 y_{23} + \alpha_7 y_{32} & \alpha_3 & \alpha_7 y_{34} + \alpha_8 y_{43} \\ \alpha_5 y_{14} + \alpha_8 y_{41} & \alpha_6 y_{24} + \alpha_8 y_{42} & \alpha_7 y_{34} + \alpha_8 y_{43} & \alpha_4 \end{pmatrix}, \quad \alpha_1, \dots, \alpha_8 \in \mathbb{R}. \quad (6)$$

Since the dimension of $\text{span } K$ is 7, the matrices at the coefficients $\alpha_1, \dots, \alpha_8$ must be linearly dependent. This is equivalent to the condition that the matrix

$$Y = \begin{pmatrix} y_{12} & y_{21} & 0 & 0 \\ y_{13} & 0 & y_{31} & 0 \\ y_{14} & 0 & 0 & y_{41} \\ 0 & y_{23} & y_{32} & 0 \\ 0 & y_{24} & 0 & y_{42} \\ 0 & 0 & y_{34} & y_{43} \end{pmatrix} \quad (7)$$

is rank-deficient, $\text{rk } Y \leq 3$. Here the rows of Y correspond to the elements $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$ of (6), respectively, and the columns to the expressions at the coefficients $\alpha_5, \dots, \alpha_8$, respectively. By construction every column of Y is nonzero.

If there exists a column of Y with exactly one nonzero element, let it be y_{ij} , then $x_ix_i^T, x_jx_j^T, x_ix_j^T + x_jx_i^T \in \text{span } K$ and K has a face which is isomorphic to \mathcal{S}_+^2 . By Lemma 8.4 the cone K is then isomorphic to either Tri_+^4 , or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or an intertwining of Han_+^3 and \mathcal{S}_+^2 .

If there exists a column of Y with exactly two nonzero elements, let them be y_{ij}, y_{ik} , then the linearly independent vectors $z_1 = x_i, z_2 = x_j, z_3 = x_k$ and scalars $\alpha = y_{ij}, \beta = y_{ik}$ satisfy the conditions of Corollary 8.3, and K is again isomorphic to one of the aforementioned cones.

Let us now assume that all elements y_{ij} for $i \neq j$ are nonzero. Then $\text{rk } Y = 3$, and the subspace spanned by the set $\{x_ix_i^T, x_iy_i^T + y_ix_i^T \mid i = 1, 2, 3, 4\}$ has dimension 7. Since this subspace is contained in $\text{span } K$, it must actually equal $\text{span } K$. There exists a nonzero vector $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ such that $Y\beta = 0$. It is easy to see that no three columns of Y can be linearly dependent, and hence all elements β_i are nonzero. By possibly multiplying y_i by the nonzero constant β_i , we may assume without loss of generality that $\beta = (1, 1, 1, 1)^T$. Then $y_{ij} = -y_{ji}$ for all $i, j = 1, \dots, 4, i \neq j$.

It is not hard to check that $\text{span } K$ can then alternatively be written as the set $\{X \in \mathcal{S}^4 \mid \langle X, Q_i \rangle = 0, i = 1, 2, 3\}$, where the linearly independent quadratic forms Q_1, Q_2, Q_3 are given by

$$\begin{pmatrix} 0 & y_{13}y_{23} & -y_{12}y_{23} & 0 \\ y_{13}y_{23} & 0 & y_{12}y_{13} & 0 \\ -y_{12}y_{23} & y_{12}y_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_{14}y_{24} & 0 & -y_{12}y_{24} \\ y_{14}y_{24} & 0 & 0 & y_{12}y_{14} \\ 0 & 0 & 0 & 0 \\ -y_{12}y_{24} & y_{12}y_{14} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & y_{14}y_{34} & -y_{13}y_{34} \\ 0 & 0 & 0 & 0 \\ y_{14}y_{34} & 0 & 0 & y_{13}y_{14} \\ -y_{13}y_{34} & 0 & y_{13}y_{14} & 0 \end{pmatrix},$$

respectively.

The rank 1 matrices in K are then given by zz^T such that $z \neq 0$ and $z^T Q_i z = 0$ for $i = 1, 2, 3$. Let us determine the set of vectors $z = (z_1, z_2, z_3, z_4)^T$ which satisfy this quadratic system of equations. It is not hard to see that if a solution z is not equal to a canonical basis vector, then all elements of z are nonzero. For such z the quadratic system can be written as

$$\begin{aligned} y_{23}^{-1} z_1^{-1} - y_{13}^{-1} z_2^{-1} + y_{12}^{-1} z_3^{-1} &= 0, \\ y_{24}^{-1} z_1^{-1} - y_{14}^{-1} z_2^{-1} + y_{12}^{-1} z_4^{-1} &= 0, \\ y_{34}^{-1} z_1^{-1} - y_{14}^{-1} z_3^{-1} + y_{13}^{-1} z_4^{-1} &= 0. \end{aligned} \tag{8}$$

This is a linear system in the unknowns z_i^{-1} . If the coefficient matrix of this system is full rank, then the solution $(z_1^{-1}, \dots, z_4^{-1})$ is proportional to $(0, y_{12}^{-1}, y_{13}^{-1}, y_{14}^{-1})$ and does not correspond to a real vector z . In this case the only rank 1 matrices in the subspace $\text{span } K$ are the matrices $x_i x_i^T$, $i = 1, \dots, 4$, and K is not ROG.

Thus the coefficient matrix of system (8) is rank deficient. This implies that all 3×3 minors of this matrix vanish, which leads to the condition $y_{14}^{-1} y_{23}^{-1} - y_{13}^{-1} y_{24}^{-1} + y_{12}^{-1} y_{34}^{-1} = 0$. The general solution of system (8) is then given by

$$\begin{pmatrix} z_1^{-1} \\ z_2^{-1} \\ z_3^{-1} \\ z_4^{-1} \end{pmatrix} = \gamma_1 \begin{pmatrix} y_{12}^{-2} + y_{13}^{-2} + y_{14}^{-2} \\ y_{13}^{-1} y_{23}^{-1} + y_{14}^{-1} y_{24}^{-1} \\ y_{14}^{-1} y_{34}^{-1} - y_{12}^{-1} y_{23}^{-1} \\ -y_{12}^{-1} y_{24}^{-1} - y_{13}^{-1} y_{34}^{-1} \end{pmatrix} + \gamma_2 \begin{pmatrix} 0 \\ y_{12}^{-1} \\ y_{13}^{-1} \\ y_{14}^{-1} \end{pmatrix}, \quad \gamma_1, \gamma_2 \in \mathbb{R}.$$

It can be checked by direct calculation that none of the 2×2 minors of the 4×2 matrix composed of the two vectors at γ_1, γ_2 , respectively, vanishes. Hence the 2-dimensional subspace of solutions of system (8) is transversal to all coordinate planes spanned by pairs of canonical basis vectors. By Lemma C.1 the cone K is then isomorphic to Han_+^4 . \square

9 Complex and quaternionic Hermitian matrices

So far we considered spectrahedral cones defined as linear sections of the cone of positive semi-definite real symmetric matrices. The definition of ROG cones can be applied also to spectrahedral cones defined as linear sections of cones of complex Hermitian or quaternionic Hermitian matrices, or even more general, as linear sections of general symmetric cones, because the rank is well-defined for the elements of these cones. We shall now consider to which extent the results developed in the preceding sections carry over to the complex and quaternionic Hermitian case, and introduce a family of complex and quaternionic Hermitian ROG cones which does not exist in the real case.

The extension of Theorem 3.18 to the case of complex or quaternionic Hermitian matrices is not straightforward and remains open. Recall that the proof of Theorem 3.18 is based on Lemma A.4, which makes an assertion about the Plücker embedding of real Grassmanians. In the case of complex or quaternionic Grassmanians, the coefficients σ_i in the formulation of this lemma have to be chosen not from the finite set $\{-1, +1\}$, but from the unit circle in the complex plane or from the quaternionic unit sphere S^3 . But then the argument at the end of the proof of Lemma A.4 is no more valid. For quaternionic matrices, the proof fails even earlier, because determinants of general quaternionic matrices and hence the Plücker embedding itself are not well-defined.

The results of Subsections 3.1 and 3.2 carry over to the case of complex or quaternionic Hermitian matrices without changes. The same holds for Lemmas 4.4, 4.7, and Corollary 4.5. Lemma 4.9 holds if we assume the direct sums in Definition 4.8 in the sense of Definition 4.2. Note, however, that the space of quaternionic vectors of length n is not a vector space. The subspaces H_i in the decomposition in Lemmas 4.7 and 4.9 have to be assumed being invariant with respect to *right* multiplication by quaternionic coefficients. The results of Subsections 4.2, 4.3, 5.1 also carry over. In the complex analog

of the construction in Subsection 5.2 we have to consider equivalence classes of quadruples of points in the complex projective plane. These are parameterized by the complex cross-ratio which leads to a family of isomorphism classes with a complex parameter. A generalization to the quaternionic case is also straightforward due to the recent development of a theory of the quaternionic cross-ratio in [9]. The complex and quaternionic analogs of the results in Subsection 6.1 are even stronger than in the real case due to the larger dimension of full faces of rank 2. For complex Hermitian matrices, every spectrahedral cone up to codimension 2 is ROG, for quaternionic Hermitian matrices up to codimension 4. Theorem 6.7 holds without changes also for the complex and quaternionic cases. The results of Section 7 can be generalized to the complex and quaternionic case with the exception of Lemma 7.13. In the quaternionic case, minimally linearly dependent sets have to be defined with respect to the multiplication by quaternionic coefficients from the *right*. The classification of complex and quaternionic simple ROG spectrahedral cones is trivial up to degree 2, for degree 3 the situation is already more complicated than in the real case.

In the complex and the quaternionic case there exists one important class of ROG spectrahedral cones which is missing in the real case, namely the positive semi-definite block-Toeplitz matrices. For the complex case this result is widely known (see, e.g., [20, Theorem 3.2]) and is equivalent to the matrix version of the Fejér-Riesz theorem [19, p.118]. Below we shall provide a proof for the quaternionic case, which is valid with appropriate modifications also for the complex case. It is based on the following spectral factorization result for hyperunitary matrices, i.e., quaternionic invertible square matrices U satisfying $U^* = U^{-1}$, where the asterisk denotes the conjugate transpose.

Lemma 9.1. *Let U be a hyperunitary matrix. Then there exists another hyperunitary matrix V of the same size and a diagonal matrix with diagonal entries on the quaternionic unit sphere S^3 such that $UV = VD$. In particular, for the columns v_1, \dots, v_n of V we have $Uv_k = v_k d_k$, where d_1, \dots, d_n are the diagonal elements of D .*

Proof. There exist a hyperunitary matrix V and an upper triangular matrix T such that $U = VTV^*$ [2]. Since U, V are hyperunitary, T must also be hyperunitary, $TT^* = I$. It follows that T is diagonal with unit norm diagonal entries. We may hence set $D = T$ and obtain $U = VDV^*$. The claim now easily follows. \square

Denote the cone of positive semi-definite Hermitian block-Toeplitz matrices consisting of $n \times n$ blocks of size $m \times m$ each by $\text{Toep}_+^{n,m}$. Recall that Hermitian block-Toeplitz matrices have the form

$$T = \begin{pmatrix} m_0 & m_1^* & \ddots & m_{n-1}^* \\ m_1 & m_0 & \ddots & m_{n-2}^* \\ \ddots & \ddots & \ddots & \ddots \\ m_{n-1} & m_{n-2} & \ddots & m_0 \end{pmatrix}, \quad (9)$$

where m_0 is a Hermitian block and m_1, \dots, m_{n-1} are general blocks of size $m \times m$. First we need a characterization of rank 1 matrices of this form.

Lemma 9.2. *A matrix of the form (9) is positive semi-definite of rank 1 if and only if there exists a non-zero quaternionic vector $v \in \mathbb{H}^m$ and a quaternion q with $|q| = 1$ such that*

$$T = \begin{pmatrix} v \\ vq \\ vq^2 \\ \vdots \\ vq^{n-1} \end{pmatrix} \begin{pmatrix} v \\ vq \\ vq^2 \\ \vdots \\ vq^{n-1} \end{pmatrix}^*. \quad (10)$$

Proof. Let T be as in (10). Since $q^* = q^{-1}$, we get $(vq^k)(vq^l)^* = vq^{k-l}v^*$, and T is of the form (9) with $m_j = vq^jv^*$.

Let now $T \in \text{Toep}_+^{n,m}$ be of rank 1. Then $T = uu^*$, where $u = (u_0^*, \dots, u_{n-1}^*)^*$ is a non-zero quaternionic vector partitioned in n subvectors of length m each. Since $m_0 = u_k u_k^*$ for all $k = 0, \dots, n-1$, every subvector has actually to be non-zero. Set $v = u_0$. Then $u_k u_k^* = vv^*$ yields

$u_k = vq_k$ for every $k = 1, \dots, n-1$, where q_k are unit norm quaternions. Set $q = q_1$. Then we have $vq_kv^* = u_ku_0^* = m_k = u_{k+1}u_1^* = vq_{k+1}q^{-1}v^*$ for all $k = 1, \dots, n-2$. This yields $q_{k+1} = q_kq$ and by induction $q_k = q^k$. \square

Lemma 9.3. *Let $T \in \text{Toep}_+^{n,m}$ be of rank N . Then T can be represented as a sum of N rank 1 matrices in $\text{Toep}_+^{n,m}$, and $\text{Toep}_+^{n,m}$ is ROG.*

Proof. There exists a $nm \times N$ matrix \tilde{W} such that $T = \tilde{W}\tilde{W}^*$. Partition \tilde{W} into blocks $\tilde{W}_0, \dots, \tilde{W}_{n-1}$ of size $m \times N$. Define $(n-1)m \times N$ matrices \tilde{W}_u, \tilde{W}_l , such that \tilde{W}_u is obtained from \tilde{W} by removal of the block \tilde{W}_{n-1} , and \tilde{W}_l is obtained by removal of the block \tilde{W}_0 . By virtue of the block-Toeplitz structure of T we have $\tilde{W}_u\tilde{W}_u^* = \tilde{W}_l\tilde{W}_l^*$. Hence there exists a $N \times N$ hyperunitary matrix U such that $\tilde{W}_l = \tilde{W}_uU$. It follows that $\tilde{W}_k = \tilde{W}_{k-1}U$ for all $k = 1, \dots, n-1$, and by iterating $\tilde{W}_k = \tilde{W}_0U^k$.

By Lemma 9.1 there exist hyperunitary matrices D, V , where D is diagonal, such that $UV = VD$. The relation $\tilde{W}_k = \tilde{W}_0U^k$ can then be rewritten as $W_k = W_0D^k$, where we have defined $W_k = \tilde{W}_kV$, $k = 0, \dots, n-1$. Define also $W = \tilde{W}V$, then we have $T = WW^*$ and W_0, \dots, W_{n-1} are the subblocks of W . Let w_1, \dots, w_N be the columns of W , then we get $T = \sum_{j=1}^N w_jw_j^*$. By virtue of the relation $W_k = W_0D^k$, each of the rank 1 matrices $w_jw_j^*$ has structure (10) with q being the j -th diagonal element of D and v the j -th column of W_0 . The application of Lemma 9.2 completes the proof. \square

10 Conclusions and open questions

In this contribution we have defined and considered a special class of spectrahedral cones, the rank 1 generated cones. These cones are characterized by Property 1.1. They have applications in optimization, namely for the approximation of difficult optimization problems by semi-definite programs, in the common case where the semi-definite program is obtained by dropping a rank 1 constraint on the matrix-valued decision variable. They are closely linked to the property of such a semi-definite relaxation being exact.

We provided many examples of ROG cones and several structural results. One of the main results has been that the geometry of a ROG cone as a convex conic subset of a real vector space uniquely determines its representation as a linear section of the positive semi-definite matrix cone, if this representation is required to satisfy Property 1.1, up to isomorphism (Theorem 3.18). In particular, every point of the cone has the same rank in every such representation. The rank also equals its Carathéodory number (Lemma 3.13). The Carathéodory number of the cone itself equals its degree as an algebraic interior (Corollary 3.14).

There exist surprisingly many ROG cones. This is due to the fact that there are several non-trivial ways to construct ROG cones of higher degree out of ROG cones of lower degree, which we have called full extensions (Subsection 4.2) and intertwining (Subsection 4.3). Besides, there is the obvious way of taking direct sums (Subsection 4.1). Iterating these procedures, one may obtain families of mutually non-isomorphic ROG cones with arbitrarily many real parameters. One may call ROG cones that are neither direct sums nor intertwining nor full extensions of other ROG cones *elementary*. Examples of elementary ROG cones are the cones of positive semi-definite block-Hankel matrices and the cones $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q \rangle = 0\}$ of codimension 1 (Subsection 6.1), where Q is an indefinite non-degenerate quadratic form. Besides these infinite series of elementary ROG cones, there exists the exceptional moment cone of the ternary quartics of dimension 15 and degree 6. It is unknown whether for the real symmetric case there exist other elementary cones.

We classified the simple ROG cones, i.e., those not representable as non-trivial direct sums, up to degree 4. There are 1,1,3,10 equivalence classes of such cones for degrees 1,2,3,4, respectively, with respect to isomorphisms.

The set of extreme rays of a ROG cone is an intersection of quadrics and hence defines a real projective variety. The varieties defined by direct sums or intertwining are finite unions of smaller projective varieties. The classification of the irreducible varieties defined by ROG cones is an open question. It would follow from a classification of the elementary ROG cones.

A Plücker embeddings of real Grassmanians

The purpose of this section is to provide Lemma A.5, which is needed for the proof of Theorem 3.18. It turns out that Lemma A.5 is essentially equivalent to a property of the Plücker embedding of real Grassmanians, which is stated below as Lemma A.4. However, we start with results on the rank 1 completion of partially specified matrices, which will be needed to prove Lemma A.4.

Definition A.1. A real partially specified $n \times m$ matrix is defined by an index subset $\mathcal{P} \subset \{1, \dots, n\} \times \{1, \dots, m\}$, called a *pattern*, together with a collection of real numbers $(A_{ij})_{(i,j) \in \mathcal{P}}$. A *completion* of a partially specified matrix $(\mathcal{P}, (A_{ij})_{(i,j) \in \mathcal{P}})$ is a real $n \times m$ matrix C such that $C_{ij} = A_{ij}$ for all $(i, j) \in \mathcal{P}$.

We shall be concerned with the question when a partially specified matrix possesses a completion of rank 1. This problem has been solved in [4], see also [10]. In order to formulate the result, we need to define a weighted bipartite graph G associated to the partially specified matrix. The two groups of vertices will be the row indices $1, \dots, n$ and the column indices $1, \dots, m$. The edges will be the elements of \mathcal{P} , with the weight of (i, j) equal to A_{ij} .

Lemma A.2. [10, Theorem 5] *A partially specified matrix $(\mathcal{P}, (A_{ij})_{(i,j) \in \mathcal{P}})$ has a rank 1 completion if and only if the following conditions are satisfied. If for some $(i, j) \in \mathcal{P}$ we have $A_{ij} = 0$, then either $A_{ij'} = 0$ for all $(i, j') \in \mathcal{P}$, or $A_{i'j} = 0$ for all $(i', j) \in \mathcal{P}$. Further, for every cycle $i_1 - j_1 - i_2 - \dots - i_k - j_k - i_1$, $k \geq 2$, of the bipartite graph G corresponding to the partially specified matrix, where i_1 in the representation of the cycle is a row index, we have $\prod_{l=1}^k A_{i_l j_l} = A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$.*

Note that the relation in the second condition of the lemma depends only on the cycle itself, but not on its starting point or on the direction in which the edges are traversed. Since the products in the lemma are multiplicative under the concatenation of paths [10, p.2171], we may also restrict the condition to prime cycles (i.e., chordless cycles where each vertex appears at most once).

Corollary A.3. *Let $A = (\mathcal{P}, (A_{ij})_{(i,j) \in \mathcal{P}})$ be a partially specified matrix such that $A_{ij} = \pm 1$ for all $(i, j) \in \mathcal{P}$, and G the corresponding bipartite graph. Assume further that for every prime cycle $i_1 - j_1 - \dots - j_k - i_1$ of G with $k \geq 2$, where the representation of the cycle begins with a row index, we have $\prod_{l=1}^k A_{i_l j_l} = A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$. Then there exists a rank 1 completion $C = ef^T$ of A such that $e \in \{-1, +1\}^n$, $f \in \{-1, +1\}^m$.*

Proof. By Lemma A.2 there exists a rank 1 completion $\tilde{C} = \tilde{e}\tilde{f}^T$ of A , where $\tilde{e} \in \mathbb{R}^n$, $\tilde{f} \in \mathbb{R}^m$. Suppose there exists an index i such that $\tilde{e}_i = 0$. Then all elements of the i -th row of \tilde{C} vanish, and all elements of this row are unspecified in A . We may then set $\tilde{e}_i = 1$ and $\tilde{e}\tilde{f}^T$ would still be a completion of A . Hence assume without loss of generality that all elements of \tilde{e} are nonzero. In a similar manner, we may assume that the elements of \tilde{f} are nonzero.

We then define the vectors $e \in \mathbb{R}^n$, $f \in \mathbb{R}^m$ element-wise by the signs of the elements of \tilde{e} , \tilde{f} , respectively. For every $(i, j) \in \mathcal{P}$ we then have $e_i f_j = \frac{\tilde{e}_i \tilde{f}_j}{|\tilde{e}_i \tilde{f}_j|} = \frac{A_{ij}}{|A_{ij}|} = A_{ij}$, because $A_{ij} = \pm 1$. It follows that $C = ef^T$ is also a completion of A . \square

We now come to the Grassmanian $Gr(n, \mathbb{R}^m)$, i.e., the space of linear n -planes in \mathbb{R}^m . Fix a basis in \mathbb{R}^m . Then an n -plane Λ can be represented by an n -tuple of linear independent vectors in \mathbb{R}^m , namely those spanning Λ . Let us treat these vectors as row vectors and stack them into an $n \times m$ matrix M . The matrix M is determined only up to left multiplication by a nonsingular $n \times n$ matrix, reflecting the ambiguity in the choice of vectors spanning Λ . The *Plücker coordinate* $\Delta_{i_1 \dots i_n}$ of Λ , where $1 \leq i_1 < \dots < i_n \leq m$, is defined as the determinant of the $n \times n$ submatrix formed of the columns i_1, \dots, i_n of M . The vector Δ of all Plücker coordinates is determined by the n -plane Λ up to multiplication by a nonzero constant and corresponds to a point in the projectivization $\mathbb{P}(\wedge^n \mathbb{R}^m)$ of the n -th exterior power of \mathbb{R}^m . The map $\Lambda \mapsto \Delta$ from $Gr(n, \mathbb{R}^m)$ to $\mathbb{P}(\wedge^n \mathbb{R}^m)$ is called the *Plücker embedding*. For a more detailed introduction into the subject see [13, Chapter 7].

Lemma A.4. *Let $\Lambda, \Lambda' \subset \mathbb{R}^m$ be two n -planes with Plücker coordinate vectors Δ, Δ' , respectively. Suppose there exists a positive constant c such that $|\Delta_{i_1 \dots i_n}| = c |\Delta'_{i_1 \dots i_n}|$ for all n -tuples (i_1, \dots, i_n) . Then there exists a linear automorphism of \mathbb{R}^m , given by a diagonal coefficient matrix $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$, where $\sigma_i \in \{-1, +1\}$ for all $i = 1, \dots, m$, which takes the n -plane Λ to Λ' .*

Proof. Assume the conditions of the lemma. Let without restriction of generality $\Delta_{1\dots n} \neq 0$, then also $\Delta'_{1\dots n} \neq 0$. Otherwise we may permute the basis vectors of \mathbb{R}^m to obtain these inequalities. Then we may choose the $n \times m$ matrix M representing Λ such that the first n columns of M form the identity matrix. Make a similar choice for the $n \times m$ matrix M' representing Λ' . Then we have $\Delta_{1\dots n} = \Delta'_{1\dots n} = 1$ and hence $c = 1$ for this choice of M, M' . If $m = n$, then we may take Σ as the identity matrix. Let $m > n$.

Let k, l be indices such that $1 \leq k \leq n, n < l \leq m$. The determinant $\Delta_{1,\dots,k-1,k+1,\dots,n,l}$ is then given by $(-1)^{n-k} M_{kl}$. Likewise, $\Delta'_{1,\dots,k-1,k+1,\dots,n,l} = (-1)^{n-k} M'_{kl}$, and hence $|M_{kl}| = |M'_{kl}|$ by the assumption on Δ, Δ' . We then get $|M_{kl}| = |M'_{kl}|$ also for all $k = 1, \dots, n, l = 1, \dots, m$.

Let now \mathcal{P} be the set of index pairs (k, l) such that $M_{kl} \neq 0$, and set $A_{kl} = \frac{M'_{kl}}{M_{kl}} \in \{-1, +1\}$ for $(k, l) \in \mathcal{P}$. Then for every completion C of the partially specified matrix $A = (\mathcal{P}, (A_{kl})_{(k,l) \in \mathcal{P}})$ we have $M' = M \bullet C$, where \bullet denotes the Hadamard matrix product.

We shall now show that the partially specified matrix A satisfies the condition of Corollary A.3. Let $i_1 - j_1 - \dots - j_k - i_1$ be a prime cycle of the bipartite graph G corresponding to A , where $k \geq 2, i_1, \dots, i_k$ are row indices, and j_1, \dots, j_k are column indices. Since the cycle is prime, the row and column indices are mutually distinct. The $k \times k$ submatrix \hat{M} of M consisting of elements with row indices i_1, \dots, i_k and column indices j_1, \dots, j_k does not have any nonzero elements except those specified by the edges of the cycle, because any such element would render the cycle non-prime. In particular, every row and every column of \hat{M} contains exactly two nonzero elements. The index set $\{j_1, \dots, j_k\}$ then has an empty intersection with $\{1, \dots, n\}$, because the first n columns of M contain strictly less than two nonzero elements each. Moreover, in the Leibniz formula for the determinant $\det \hat{M}$ only two products are nonzero, and the corresponding permutations are related by a cyclic permutation, which has sign $(-1)^{k-1}$. Therefore we have $|\det \hat{M}| = \left| \prod_{l=1}^k M_{i_l j_l} - (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l} \right|$.

Consider the $n \times n$ submatrix of M consisting of columns with indices in $(\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}) \cup \{j_1, \dots, j_k\}$. The determinant of this submatrix has absolute value $|\det \hat{M}|$ by construction. A similar formula holds for the absolute value of the determinant of the corresponding $n \times n$ submatrix of M' . By the assumption on Δ, Δ' we then have

$$\left| \prod_{l=1}^k M_{i_l j_l} - (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l} \right| = \left| \prod_{l=1}^k M'_{i_l j_l} - (-1)^k M'_{i_k j_1} \cdot \prod_{l=1}^{k-1} M'_{i_{l+1} j_l} \right|.$$

It follows that either

$$\left(1 - \prod_{l=1}^k A_{i_l j_l} \right) \prod_{l=1}^k M_{i_l j_l} = \left(1 - A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l} \right) (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l}$$

or

$$\left(1 + \prod_{l=1}^k A_{i_l j_l} \right) \prod_{l=1}^k M_{i_l j_l} = \left(1 + A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l} \right) (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l}.$$

Note that all the involved elements of M are nonzero, while those of A equal ± 1 . The relation $\prod_{l=1}^k A_{i_l j_l} = -A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$ would then imply that in each of the two equations above, one side is zero while the other is not. Therefore we must have $\prod_{l=1}^k A_{i_l j_l} = A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$, and the condition in Corollary A.3 is fulfilled.

By this corollary there exists a rank 1 completion $C = ef^T$ of A such that $e \in \{-1, +1\}^n, f \in \{-1, +1\}^m$. We then have $M' = M \bullet (ef^T) = \text{diag}(e) \cdot M \cdot \text{diag}(f)$. Setting $\Sigma = \text{diag}(f)$ completes the proof. \square

We now provide the technical result which is necessary for the proof of Theorem 3.18.

Lemma A.5. *Let $x_1, \dots, x_m, y_1, \dots, y_m \in \mathbb{R}^n$ be such that $\text{span}\{x_1, \dots, x_m\} = \text{span}\{y_1, \dots, y_m\} = \mathbb{R}^n$. Denote by $L \subset \mathcal{S}^n$ the linear span of the set $\{x_1 x_1^T, \dots, x_m x_m^T\}$ and assume there exists a linear map $\tilde{f} : L \rightarrow \mathcal{S}^n$ such that $\tilde{f}(x_i x_i^T) = y_i y_i^T$ for all $i = 1, \dots, m$. Assume further that there exists a positive constant $c > 0$ such that $\det Z = c \det \tilde{f}(Z)$ for all $Z \in L$. Then there exists a non-singular $n \times n$ matrix S such that $\tilde{f}(Z) = SZS^T$ for all matrices $Z \in L$.*

Proof. Assemble the column vectors x_i into an $n \times m$ matrix X and the column vectors y_i into an $n \times m$ matrix Y . By assumption these matrices have full row rank n . For mutually distinct indices $i_1, \dots, i_n \in$

$\{1, \dots, m\}$, let $X_{i_1 \dots i_n}, Y_{i_1 \dots i_n}$ be the $n \times n$ submatrices formed of the columns i_1, \dots, i_n of X, Y , respectively. We have $\det(X_{i_1 \dots i_n} X_{i_1 \dots i_n}^T) = \det(\sum_{k=1}^n x_{i_k} x_{i_k}^T) = c \det(\sum_{k=1}^n y_{i_k} y_{i_k}^T) = c \det(Y_{i_1 \dots i_n} Y_{i_1 \dots i_n}^T)$, which implies $|\det X_{i_1 \dots i_n}| = \sqrt{c} |\det Y_{i_1 \dots i_n}|$.

Since the n -tuple (i_1, \dots, i_n) was chosen arbitrarily, the n -planes spanned in \mathbb{R}^m by the row vectors of X, Y , respectively, fulfill the conditions of Lemma A.4. By this lemma there exist a nonsingular $n \times n$ matrix S and a diagonal matrix $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$ with $\sigma_i \in \{-1, +1\}$ such that $Y = SX\Sigma$, or equivalently $y_i = \sigma_i S x_i$ for all $i = 1, \dots, m$. For every $i = 1, \dots, m$ we then have $\tilde{f}(x_i x_i^T) = y_i y_i^T = S x_i x_i^T S^T$, and by linear extension we get the claim of the lemma. \square

B Extreme elements of rank 2

In this section we provide auxiliary results which are needed for the classification of ROG spectrahedral cones of codimension 2 in Subsection 6.2. By virtue of Lemma 6.2 these results allow in principle also a classification of ROG cones of codimensions 3 and 4, but the number and complexity of cases to be considered becomes prohibitive in the framework of this paper.

We first provide the following structural result on real symmetric matrix pencils. Recall that y is called an eigenvector of the pencil $Q_1 + \lambda Q_2$ if the linear forms $Q_1 y, Q_2 y$ are linearly dependent.

Lemma B.1. *Let Q_1, Q_2 be quadratic forms on \mathbb{R}^n such that the pencil $Q_1 + \lambda Q_2$ possesses n linearly independent real eigenvectors. Then there exists a direct sum decomposition $\mathbb{R}^n = H_0 \oplus H_1 \oplus \dots \oplus H_m$, non-degenerate quadratic forms Φ_k on H_k , $k = 1, \dots, m$, and mutually distinct angles $\varphi_1, \dots, \varphi_m \in [0, \pi)$ with the following properties. For every vector $x = \sum_{k=0}^m x_k$, where $x_k \in H_k$, we have $Q_1(x) = \sum_{k=1}^m \cos \varphi_k \Phi_k(x_k)$, $Q_2(x) = \sum_{k=1}^m \sin \varphi_k \Phi_k(x_k)$. Moreover, the set of real eigenvectors of the pencil $Q_1 + \lambda Q_2$ is given by the union $\bigcup_{k=1}^m (H_0 + H_k)$.*

Proof. We define the subspace H_0 as the intersection $\ker Q_1 \cap \ker Q_2$. For every real eigenvector $y \notin H_0$ of the pencil $Q_1 + \lambda Q_2$, the linear span of the set $\{Q_1 y, Q_2 y\}$ of linear forms has then dimension 1. Hence there exists a unique angle $\varphi(y) \in [0, \pi)$ such that $\sin \varphi(y) Q_1 y - \cos \varphi(y) Q_2 y = 0$.

By assumption we find linearly independent real eigenvectors $y_1, \dots, y_{n-\dim H_0}$ of the pencil $Q_1 + \lambda Q_2$ such that $\text{span}(H_0 \cup \{y_1, \dots, y_{n-\dim H_0}\}) = \mathbb{R}^n$. Regroup these vectors into subsets $\{y_{11}, \dots, y_{1d_1}\}, \dots, \{y_{m1}, \dots, y_{md_m}\}$ such that $\varphi(y_{kl}) = \varphi_k$, $k = 1, \dots, m$, $l = 1, \dots, d_k$, where $\varphi_1, \dots, \varphi_m \in [0, \pi)$ are mutually distinct angles, and d_k is the number of eigenvectors corresponding to angle φ_k . Define the subspace H_k as the linear span of y_{k1}, \dots, y_{kd_k} , $k = 1, \dots, m$. Then by construction we have that $H_0 \oplus H_1 \oplus \dots \oplus H_m$ is a direct sum decomposition of \mathbb{R}^n . Moreover, every vector $y \in H_k$ is an eigenvector and we have $\sin \varphi_k Q_1 y - \cos \varphi_k Q_2 y = 0$ for all $y \in H_k$, $k = 1, \dots, m$. It follows that there exist quadratic forms Φ_k on H_k , $k = 1, \dots, m$, such that $Q_1|_{H_k} = \cos \varphi_k \Phi_k$, $Q_2|_{H_k} = \sin \varphi_k \Phi_k$.

Let now $k, k' \in \{1, \dots, m\}$ be distinct indices and $y \in H_k$, $y' \in H_{k'}$ be arbitrary vectors. By construction we have $\sin \varphi_k Q_1 y - \cos \varphi_k Q_2 y = \sin \varphi_{k'} Q_1 y' - \cos \varphi_{k'} Q_2 y' = 0$. Therefore $\sin \varphi_k y^T Q_1 y' - \cos \varphi_k y^T Q_2 y' = \sin \varphi_{k'} y'^T Q_1 y - \cos \varphi_{k'} y'^T Q_2 y = 0$. But $\varphi_k, \varphi_{k'}$ are distinct, and thus this linear system on $y^T Q_i y'$ has only the trivial solution $y^T Q_1 y' = y^T Q_2 y' = 0$. The decomposition formulas $Q_1(x) = \sum_{k=1}^m \cos \varphi_k \Phi_k(x_k)$, $Q_2(x) = \sum_{k=1}^m \sin \varphi_k \Phi_k(x_k)$ now readily follow.

Let $1 \leq k \leq m$. Suppose there exists a vector $y \in H_k$ such that $\Phi_k y = 0$. Then we have $Q_1 y = Q_2 y = 0$, and $y \in H_0$. Thus $y = 0$, and the form Φ_k must be non-degenerate.

Let now $x = \sum_{k=0}^m x_k$ be a real eigenvector of the pencil $Q_1 + \lambda Q_2$, where $x_k \in H_k$. Then we have $\sin \varphi Q_1 x - \cos \varphi Q_2 x = 0$ for some angle $\varphi \in [0, \pi)$. Let $z_k \in H_k$, $k = 0, \dots, m$ be arbitrary vectors, and set $z = \sum_{k=0}^m z_k$. Then we have $x^T Q_1 z = \sum_{k=1}^m \cos \varphi_k \Phi_k(x_k, z_k)$, $x^T Q_2 z = \sum_{k=1}^m \sin \varphi_k \Phi_k(x_k, z_k)$. It follows that

$$\begin{aligned} 0 &= \sin \varphi x^T Q_1 z - \cos \varphi x^T Q_2 z = \sum_{k=1}^m (\sin \varphi \cos \varphi_k \Phi_k(x_k, z_k) - \cos \varphi \sin \varphi_k \Phi_k(x_k, z_k)) \\ &= \sum_{k=1}^m \sin(\varphi - \varphi_k) \Phi_k(x_k, z_k). \end{aligned}$$

Here $\Phi_k(x_k, z_k) = \frac{1}{4}(\Phi_k(x_k + z_k) - \Phi_k(x_k - z_k))$ is as usual the bilinear form defined by the quadratic form Φ_k . Since this holds identically for all $z_k \in H_k$ and the forms Φ_k are non-degenerate, we must have either $\varphi = \varphi_k$ or $x_k = 0$ for each $k = 1, \dots, m$. Therefore $x \in H_0 + H_k$ for some k . On the

other hand, every vector $x \in H_0 + H_k$ is an eigenvector of the pencil $Q_1 + \lambda Q_2$, since it satisfies $\sin \varphi_k Q_1 x - \cos \varphi_k Q_2 x = 0$. \square

We now come to spectrahedral cones $K \subset \mathcal{S}_+^n$ of codimension d . We represent these as in Lemma 6.1 by linearly independent quadratic forms Q_1, \dots, Q_d on \mathbb{R}^n , $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_i \rangle = 0, i = 1, \dots, d\}$. We study the intersections of the spectrahedral cone K with faces $\mathcal{F}_n(H)$ of \mathcal{S}_+^n of rank not exceeding 2, i.e., where $H = \text{span}\{x, y\}$ for some vectors $x, y \in \mathbb{R}^n$.

Lemma B.2. *Assume above notations. The face $\mathcal{F}_n(H) \cap K$ of K is generated by an extreme element of rank 2 if and only if the $d \times 3$ matrix*

$$M(x, y) = \begin{pmatrix} x^T Q_1 x & 2x^T Q_1 y & y^T Q_1 y \\ \vdots & \vdots & \vdots \\ x^T Q_d x & 2x^T Q_d y & y^T Q_d y \end{pmatrix}$$

has rank 2 and its kernel is generated by a vector $(a, b, c)^T \in \mathbb{R}^3$ such that the matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is definite.

Proof. If x, y are linearly dependent, then both the face $\mathcal{F}_n(H)$ and the matrix $M(x, y)$ have rank at most 1. Hence we may assume that x, y are linearly independent.

The intersection $\mathcal{F}_n(H) \cap K$ is given by all matrices $X = axx^T + b(xy^T + yx^T) + cyy^T$ such that $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0$ and

$$\begin{pmatrix} \langle X, Q_1 \rangle \\ \vdots \\ \langle X, Q_d \rangle \end{pmatrix} = M(x, y) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.$$

The signature of $X = axx^T + b(xy^T + yx^T) + cyy^T$ equals the signature of A .

Suppose that $\text{rk } M(x, y) = 2$ and the kernel of $M(x, y)$ is generated by a vector $(a, b, c)^T$ such that $A \succ 0$. Then $X = axx^T + b(xy^T + yx^T) + cyy^T$ is positive semi-definite of rank 2, and the face $\mathcal{F}_n(H) \cap K$ is generated by X , which proves the "if" direction.

Suppose now that $X = axx^T + b(xy^T + yx^T) + cyy^T$ is positive semi-definite of rank 2 and generates the face $\mathcal{F}_n(H) \cap K$. Then $(a, b, c)^T \in \ker M(x, y)$ and $A \succ 0$. Moreover, the dimension of $\ker M(x, y)$ is 1 and it must be generated by the vector $(a, b, c)^T$, otherwise we would have $\dim(\mathcal{F}_n(H) \cap K) > 1$. It follows that $\text{rk } M(x, y) = 2$, which proves the "only if" direction. \square

Lemma B.3. *Assume the notations of the previous lemma and set $d = 2$. The matrix $M(x, y)$ satisfies the conditions of the previous lemma if and only if the bi-quartic polynomial $p(x, y)$ given by*

$$(y^T Q_1 y \cdot x^T Q_2 x - x^T Q_1 x \cdot y^T Q_2 y)^2 - 4(x^T Q_1 y \cdot y^T Q_2 y - x^T Q_2 y \cdot y^T Q_1 y)(x^T Q_1 x \cdot x^T Q_2 y - x^T Q_1 y \cdot x^T Q_2 x)$$

is negative on x, y .

Proof. The matrix A is definite if and only if $b^2 - ac < 0$, and $M(x, y)$ has full rank 2 if and only if the cross product

$$\begin{pmatrix} x^T Q_1 x \\ 2x^T Q_1 y \\ y^T Q_1 y \end{pmatrix} \times \begin{pmatrix} x^T Q_2 x \\ 2x^T Q_2 y \\ y^T Q_2 y \end{pmatrix} = \begin{pmatrix} 2(x^T Q_1 y \cdot y^T Q_2 y - x^T Q_2 y \cdot y^T Q_1 y) \\ y^T Q_1 y \cdot x^T Q_2 x - x^T Q_1 x \cdot y^T Q_2 y \\ 2(x^T Q_1 x \cdot x^T Q_2 y - x^T Q_1 y \cdot x^T Q_2 x) \end{pmatrix}$$

is nonzero. In this case the kernel of $M(x, y)$ is generated by this cross product, and hence $b^2 - ac < 0$ if and only if $p(x, y) < 0$. \square

Lemma B.4. *Assume the notations of the previous lemma and suppose that the polynomial $p(x, y)$ is nonnegative for all $x, y \in \mathbb{R}^n$. Suppose there exists $z \in \mathbb{R}^n$ such that $z^T Q_1 z = z^T Q_2 z = 0$ and the linear forms $q_1 = Q_1 z$, $q_2 = Q_2 z$ are linearly independent. Then there exists a linear form u which is linearly independent from q_1, q_2 and such that $Q_1 = u \otimes q_1 + q_1 \otimes u$, $Q_2 = u \otimes q_2 + q_2 \otimes u$.*

Proof. By virtue of the condition $z^T Q_1 z = z^T Q_2 z = 0$ the nonnegative polynomial $p(x, y)$ vanishes for $x = z$ and all $y \in \mathbb{R}^n$. Therefore $\left. \frac{\partial p(x, y)}{\partial x} \right|_{x=z} = 0$ for all $y \in \mathbb{R}^n$. By virtue of $z^T Q_1 z = z^T Q_2 z = 0$, at $x = z$ this gradient is given by

$$\left. \frac{\partial p(x, y)}{\partial x} \right|_{x=z} = -8(q_1^T y \cdot y^T Q_2 y - q_2^T y \cdot y^T Q_1 y)(q_2^T y \cdot q_1 - q_1^T y \cdot q_2) = 0.$$

Since q_1, q_2 are linearly independent, the linear form $q_2^T y \cdot q_1 - q_1^T y \cdot q_2$ is nonzero if $q_1^T y \neq 0$ or $q_2^T y \neq 0$. Therefore $q_1^T y \cdot y^T Q_2 y = q_2^T y \cdot y^T Q_1 y$ for all such y , i.e., for a dense subset of \mathbb{R}^n . It follows that $q_1^T y \cdot y^T Q_2 y = q_2^T y \cdot y^T Q_1 y$ identically for all $y \in \mathbb{R}^n$.

In particular, for every $y \in \mathbb{R}^n$ such that $q_1^T y = 0$, $q_2^T y \neq 0$ we have $y^T Q_1 y = 0$. The subset of such vectors y is dense in the kernel of q_1 , and hence Q_1 is zero on this kernel. It follows that there exists a linear form u_1 such that $Q_1 = q_1 \otimes u_1 + u_1 \otimes q_1$. In a similar manner, there exists a linear form u_2 such that $Q_2 = q_2 \otimes u_2 + u_2 \otimes q_2$. It follows that $q_1^T y \cdot q_2^T y \cdot u_2^T y = q_2^T y \cdot q_1^T y \cdot u_1^T y$ identically for all $y \in \mathbb{R}^n$. For all $y \in \mathbb{R}^n$ such that $q_1^T y \neq 0$ and $q_2^T y \neq 0$ it follows that $u_2^T y = u_1^T y$. Since the set of such vectors y is dense in \mathbb{R}^n , we get that u_1, u_2 are equal to the same linear form u .

Note that $q_1^T z = q_2^T z = 0$ by assumption. We obtain $q_1 = Q_1 z = q_1^T z \cdot u + u^T z \cdot q_1 = u^T z \cdot q_1$, and hence $u^T z = 1$. Therefore u must be linearly independent of q_1, q_2 . \square

Lemma B.5. *Let Q_1, Q_2 be linearly independent quadratic forms on \mathbb{R}^n . Consider the spectrahedral cone $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$. Suppose that K does not have extreme elements of rank 2. Then one of the following conditions holds.*

(i) *For every $z \in \mathbb{R}^n$ such that $z^T Q_1 z = z^T Q_2 z = 0$ the linear forms $q_1 = Q_1 z$, $q_2 = Q_2 z$ are linearly dependent.*

(ii) *There exist linearly independent linear forms q_1, q_2, u such that $Q_1 = u \otimes q_1 + q_1 \otimes u$, $Q_2 = u \otimes q_2 + q_2 \otimes u$.*

Proof. Suppose that condition (i) does not hold, then there exists $z \in \mathbb{R}^n$ such that $z^T Q_1 z = z^T Q_2 z = 0$ and the linear forms $q_1 = Q_1 z$, $q_2 = Q_2 z$ are linearly independent. Further, by Lemmas B.2, B.3 the polynomial $p(x, y)$ defined in Lemma B.3 is nonnegative for all $x, y \in \mathbb{R}^n$. Hence the conditions of Lemma B.4 are fulfilled and condition (ii) holds. \square

C Coordinate-free characterization of Han_+^4

In this section we provide an auxiliary result which is necessary for the classification of simple ROG cones of degree 4 and dimension 7.

Lemma C.1. *Let e_1, \dots, e_4 be the canonical basis vectors of \mathbb{R}^4 . Let $P \subset \mathbb{R}^4$ be a 2-dimensional subspace which is transversal to all coordinate planes spanned by pairs of basis vectors. Define the set of vectors*

$$\begin{aligned} \mathcal{R} &= \{\alpha e_i \mid \alpha \in \mathbb{R}, i = 1, 2, 3, 4\} \cup \\ &\quad \{z = (z_1, z_2, z_3, z_4)^T \in \mathbb{R}^4 \mid z_i \neq 0 \ \forall i = 1, \dots, 4; (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1})^T \in P\}. \end{aligned}$$

Let $L \subset \mathcal{S}_+^4$ be the linear span of the set $\{zz^T \mid z \in \mathcal{R}\}$. Then $\dim L = 7$, and the spectrahedral cone $K = L \cap \mathcal{S}_+^4$ is isomorphic to the cone Han_+^4 of positive semi-definite 4×4 Hankel matrices.

Proof. Let $(r_1 \cos \varphi_1, \dots, r_4 \cos \varphi_4)^T, (r_1 \sin \varphi_1, \dots, r_4 \sin \varphi_4)^T \in \mathbb{R}^4$ be two linearly independent vectors spanning P . By the transversality property of P all 2×2 minors of the 4×2 matrix composed of these vectors are nonzero. Hence the angles $\varphi_1, \dots, \varphi_4$ are mutually distinct modulo π , and the scalars r_1, \dots, r_4 are nonzero. We may also assume without loss of generality that none of the angles φ_i is a multiple of π , otherwise we choose slightly different basis vectors in P .

For all $\xi \in [0, \pi)$ we then have that the vector $(r_1 \sin(\varphi_1 + \xi), \dots, r_4 \sin(\varphi_4 + \xi))^T$ is an element of P . More precisely, we get

$$\mathcal{R} = \{\alpha e_i \mid \alpha \in \mathbb{R}, i = 1, 2, 3, 4\} \cup \left\{ \alpha \left(\frac{1}{r_1 \sin(\varphi_1 + \xi)}, \dots, \frac{1}{r_4 \sin(\varphi_4 + \xi)} \right)^T \mid \alpha \in \mathbb{R}, \xi \neq \varphi_i \pmod{\pi} \right\}.$$

Now set $\cos \xi = \frac{1-t^2}{1+t^2}$, $\sin \xi = \frac{2t}{1+t^2}$, and $s = t - \frac{1}{t}$. Then $\frac{1}{r_i \sin(\varphi_i + \xi)} = \frac{1+t^2}{r_i t(2 \cos \varphi_i - s \sin \varphi_i)}$. Define the vector $\mu(s) = \left(\frac{1}{r_1(2 \cos \varphi_1 - s \sin \varphi_1)}, \dots, \frac{1}{r_4(2 \cos \varphi_4 - s \sin \varphi_4)} \right)^T$ for all $s \in \mathbb{R}$ except the values $s = 2 \cot \varphi_i$, $i = 1, \dots, 4$. We then get

$$\begin{aligned} \mathcal{R} &= \{ \alpha e_i \mid \alpha \in \mathbb{R}, i = 1, 2, 3, 4 \} \cup \{ \alpha \mu(s) \mid \alpha \in \mathbb{R}, s \in \mathbb{R}, s \neq 2 \cot \varphi_i \} \cup \\ &\cup \left\{ \alpha \left(\frac{1}{r_1 \sin \varphi_1}, \dots, \frac{1}{r_4 \sin \varphi_4} \right)^T \mid \alpha \in \mathbb{R} \right\}. \end{aligned}$$

Multiplying the vector $\mu(s)$ by the common denominator of its elements, we obtain the vector

$$\nu(s) = \mu(s) \cdot \prod_{i=1}^4 (2 \cos \varphi_i - s \sin \varphi_i) = \text{diag}(r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) \cdot M \cdot \text{diag}(8, -4, 2, -1) \cdot \eta(s), \quad (11)$$

where $\eta(s) = (1, s, s^2, s^3)^T$ and the matrix M is given by

$$\begin{pmatrix} \cos \varphi_2 \cos \varphi_3 \cos \varphi_4 & \sin \varphi_2 \sin \varphi_3 \sin \varphi_4 + \sin(\varphi_2 + \varphi_3 + \varphi_4) & \cos \varphi_2 \cos \varphi_3 \cos \varphi_4 - \cos(\varphi_2 + \varphi_3 + \varphi_4) & \sin \varphi_2 \sin \varphi_3 \sin \varphi_4 \\ \cos \varphi_1 \cos \varphi_3 \cos \varphi_4 & \sin \varphi_1 \sin \varphi_3 \sin \varphi_4 + \sin(\varphi_1 + \varphi_3 + \varphi_4) & \cos \varphi_1 \cos \varphi_3 \cos \varphi_4 - \cos(\varphi_1 + \varphi_3 + \varphi_4) & \sin \varphi_1 \sin \varphi_3 \sin \varphi_4 \\ \cos \varphi_1 \cos \varphi_2 \cos \varphi_4 & \sin \varphi_1 \sin \varphi_2 \sin \varphi_4 + \sin(\varphi_1 + \varphi_2 + \varphi_4) & \cos \varphi_1 \cos \varphi_2 \cos \varphi_4 - \cos(\varphi_1 + \varphi_2 + \varphi_4) & \sin \varphi_1 \sin \varphi_2 \sin \varphi_4 \\ \cos \varphi_1 \cos \varphi_2 \cos \varphi_3 & \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 + \sin(\varphi_1 + \varphi_2 + \varphi_3) & \cos \varphi_1 \cos \varphi_2 \cos \varphi_3 - \cos(\varphi_1 + \varphi_2 + \varphi_3) & \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \end{pmatrix}.$$

Here for the calculus of M we used the formulas

$$\begin{aligned} \sin \varphi_i \cos \varphi_j \cos \varphi_k + \sin \varphi_j \cos \varphi_i \cos \varphi_k + \sin \varphi_k \cos \varphi_i \cos \varphi_j &= \sin \varphi_i \sin \varphi_j \sin \varphi_k + \sin(\varphi_i + \varphi_j + \varphi_k), \\ \sin \varphi_2 \sin \varphi_3 \cos \varphi_4 + \sin \varphi_2 \sin \varphi_4 \cos \varphi_3 + \sin \varphi_3 \sin \varphi_4 \cos \varphi_2 &= \cos \varphi_i \cos \varphi_j \cos \varphi_k - \cos(\varphi_i + \varphi_j + \varphi_k). \end{aligned}$$

Note that the vector $\nu(s)$ can also be defined by the right-hand side of (11) for $s = 2 \cot \varphi_i$, and for this value of s it is proportional to e_i . Defining $\eta(\infty) = e_4$ and $\nu(\infty) = \text{diag}(r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) \cdot M \cdot \text{diag}(8, -4, 2, -1) \cdot \eta(\infty)$, we finally get

$$\mathcal{R} = \{ \alpha \nu(s) \mid \alpha \in \mathbb{R}, s \in \mathbb{R} \cup \{\infty\} \}.$$

A symbolic computation with a computer algebra system yields

$$\det M = \sin(\varphi_1 - \varphi_2) \sin(\varphi_1 - \varphi_3) \sin(\varphi_1 - \varphi_4) \sin(\varphi_2 - \varphi_3) \sin(\varphi_2 - \varphi_4) \sin(\varphi_3 - \varphi_4) \neq 0.$$

Hence the matrix product $\text{diag}(r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) \cdot M \cdot \text{diag}(8, -4, 2, -1)$ is non-degenerate, and the subspace $L = \text{span}\{zz^T \mid z = \nu(s), s \in \mathbb{R} \cup \{\infty\}\}$ is isomorphic to the subspace $L' = \text{span}\{zz^T \mid z = \eta(s), s \in \mathbb{R} \cup \{\infty\}\}$.

The subspace L' , however, is the subspace of Hankel matrices in \mathcal{S}^4 . The claim of the lemma now easily follows. \square

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